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–	Day 1
1	Let ABC be an acute scalene triangle such that $AB < AC$. The midpoints of sides AB and AC are M and N, respectively. Let P and Q be points on the line MN such that $\angle CBP = \angle ACB$ and $\angle QCB = \angle CBA$. The circumscribed circle of triangle ABP intersects line AC at D ($D \neq A$) and the circumscribed circle of triangle AQC intersects line AB at E ($E \neq A$). Show that lines $BC, DP,$ and EQ are concurrent. Nicols De la Hoz, Colombia
2	Let T_n denotes the least natural such that $$n \mid 1+2+3+\cdots+T_n=\sum_{i=1}^{T_n} i$$ Find all naturals m such that $m \geq T_m$. *Proposed by Nicols De la Hoz*
3	Let $n \geq 2$ be an integer. A sequence $\alpha = (a_1, a_2, ..., a_n)$ of n integers is called *Lima* if $\gcd\{a_i - a_j$ such that $a_i > a_j$ and $1 \leq i, j \leq n\} = 1$, that is, if the greatest common divisor of all the differences $a_i - a_j$ with $a_i > a_j$ is 1. One operation consists of choosing two elements a_k and a_ℓ from a sequence, with $k \neq \ell$, and replacing a_ℓ by $a'_\ell = 2a_k - a_\ell$. Show that, given a collection of $2^n - 1$ Lima sequences, each one formed by n integers, there are two of them, say β and γ, such that it is possible to transform β into γ through a finite number of operations. Notes. The sequences $(1, 2, 2, 7)$ and $(2, 7, 2, 1)$ have the same elements but are different. If all the elements of a sequence are equal, then that sequence is not Lima.
–	Day 2
4	Show that there exists a set $\mathcal{C}$ of 2020 distinct, positive integers that satisfies simultaneously the following properties: • When one computes the greatest common divisor of each pair of elements of $\mathcal{C}$, one gets a list of numbers that are all distinct. • When one computes the least common multiple of each pair of elements of $\mathcal{C}$, one gets a list of numbers that are all distinct.

5 Determine all functions $f:\mathbb{R}\rightarrow \mathbb{R}$ such that

$$f(xf(x-y))+yf(x)=x+y+f(x^2),$$

for all real numbers x and y.

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6 Let ABC be an acute, scalene triangle. Let H be the orthocenter and O be the circumcenter of triangle ABC, and let P be a point interior to the segment HO. The circle with center P and radius PA intersects the lines AB and AC again at R and S, respectively. Denote by Q the symmetric point of P with respect to the perpendicular bisector of BC. Prove that points P, Q, R and S lie on the same circle.
