Estonia Team Selection Test 2020

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–	Day 1
1	Let ABC be a triangle. Circle Γ passes through A, meets segments AB and AC again at points D and E respectively, and intersects segment BC at F and G such that F lies between B and G. The tangent to circle BDF at F and the tangent to circle CEG at G meet at point T. Suppose that points A and T are distinct. Prove that line AT is parallel to BC. (Nigeria)
2	There are 2020 inhabitants in a town. Before Christmas, they are all happy; but if an inhabitant does not receive any Christmas card from any other inhabitant, he or she will become sad. Unfortunately, there is only one post company which offers only one kind of service: before Christmas, each inhabitant may appoint two different other inhabitants, among which the company chooses one to whom to send a Christmas card on behalf of that inhabitant. It is known that the company makes the choices in such a way that as many inhabitants as possible will become sad. Find the least possible number of inhabitants who will become sad.
3	Find all functions $f : R \to R$ such that for all real numbers x and y $$f(x^3+y^3)=f(x^3)+3x^3f(x)f(y)+3f(x)(f(y))^2+y^6f(y)$$
–	Day 2
1	For every positive integer x, let $k(x)$ denote the number of composite numbers that do not exceed x. Find all positive integers n for which $(k(n))!$ lcm $(1, 2,..., n) > (n-1)!$.
2	The radius of the circumcircle of triangle Δ is R and the radius of the inscribed circle is r. Prove that a circle of radius $R + r$ has an area more than 5 times the area of triangle Δ.
3	With expressions containing the symbol $*$, the following transformations can be performed: 1) rewrite the expression in the form $x * (y * z) as ((1 * x) * y) * z$; 2) rewrite the expression in the form $x * 1$ as x. Conversions can only be performed with an integer expression, but not with its parts. For example, $(1 * 1) * (1 * 1)$ can be rewritten according to the first rule as $((1 * (1 * 1)) * 1) * 1$ (taking $x = 1 * 1$, $y = 1$ and $z = 1$), but not as $1 * (1 * 1)$ or $(1 * 1) * 1$ (in the last two cases, the

second rule would be applied separately to the left or right side $1 * 1$).
Find all positive integers n for which the expression $\underbrace{1 * (1 * (1 * (... * (1 * 1)...))}_{nunits}$ it is possible to lead to a form in which there is not a single asterisk.

Note. The expressions $(x * y)*$z and $x * (y * z)$ are considered different, also, in the general case, the expressions $x * y$ and $y * x$ are different.

– Day 3

1 The infinite sequence $a_0, a_1, a_2, \dots$ of (not necessarily distinct) integers has the following properties: $0 \le a_i \le i$ for all integers $i \ge 0$, and

$$\binom{k}{a_0} + \binom{k}{a_1} + \cdots + \binom{k}{a_k} = 2^k$$

for all integers $k \ge 0$. Prove that all integers $N \ge 0$ occur in the sequence (that is, for all $N \ge 0$, there exists $i \ge 0$ with $a_i = N$).

2 Let M be the midpoint of side BC of an acute-angled triangle ABC. Let D and E be the center of the excircle of triangle AMB tangent to side AB and the center of the excircle of triangle AMC tangent to side AC, respectively. The circumscribed circle of triangle ABD intersects lineBC for the second time at point F, and the circumcircle of triangle ACE is at point G. Prove that $|BF| = |CG|$.

3 We say that a set S of integers is *rootiful* if, for any positive integer n and any $a_0, a_1, \cdots, a_n \in S$, all integer roots of the polynomial $a_0 + a_1 x + \cdots + a_n x^n$ are also in S. Find all rootiful sets of integers that contain all numbers of the form $2^a - 2^b$ for positive integers a and b.

– Day 4

1 Let $a_1, a_2, ...$ a sequence of real numbers.
For each positive integer n, we denote $m_n = \frac{a_1 + a_2 + ... + a_n}{n}$.
It is known that there exists a real number c such that for any different positive integers i, j, k: $(i - j)m_k + (j - k)m_i + (k - i)m_j = c$.
Prove that the sequence $a_1, a_2, ..$ is arithmetic

2 Let n be an integer, $n \ge 3$. Select n points on the plane, none of which are three on the same line. Consider all triangles with vertices at selected points, denote the smallest of all the interior angles of these triangles by the variable α. Find the largest possible value of α and identify all the selected n point placements for which the max occurs.

3 The prime numbers p and q and the integer a are chosen such that $p > 2$ and $a \not\equiv 1$ (mod q), but $a^p \equiv 1$ (mod q). Prove that $(1 + a^1)(1 + a^2)...(1 + a^{p-1}) \equiv 1$ (mod q) .