National Math Olympiad (3rd Round) 2020
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– Algebra

1 find all functions from the reals to themselves. such that for every real x, y.

$$f(y-f(x)) = f(x) - 2x + f(f(y))$$

2 let $a_1, a_2, ..., a_n$,$b_1, b_2, ..., b_n$,$c_1, c_2, ..., c_n$ be real numbers. prove that

$$\sum_{cyc} \sqrt{\sum_{i\in\{1,...,n\}} (3a_i - b_i - c_i)^2} \geq \sum_{cyc} \sqrt{\sum_{i\in\{1,2,...,n\}} a_i^2}$$

3 find all k distinct integers $a_1, a_2, ..., a_k$ such that there exists an injective function f from reals to themselves such that for each positive integer n we have

$$\{f^n(x) - x | x \in \mathbb{R}\} = \{a_1 + n, a_2 + n, ..., a_k + n\}$$

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4 We call a polynomial $P(x)$ intresting if there are 1398 distinct positive integers $n_1, ..., n_{1398}$ such that

$$P(x) = \sum x^{n_i} + 1$$

Does there exist infinitly many polynomials $P_1(x), P_2(x), ...$ such that for each distinct i, j the polynomial $P_i(x)P_j(x)$ is interesting.

– Geometry

1 Let $ABCD$ be a Rhombus and let w be it's incircle. Let M be the midpoint of AB the point K is on w and inside $ABCD$ such that MK is tangent to w. Prove that $CDKM$ is cyclic.

2 Triangle ABC with it's circumcircle Γ is given. Points D and E are chosen on segment BC such that $\angle BAD = \angle CAE$. The circle ω is tangent to AD at A with it's circumcenter lies on Γ. Reflection of A through BC is A'. If the line $A'E$ meet ω at L and K. Then prove either BL and CK or BK and CL meet on Γ.

3 The circle Ω with center I_A, is the A-excircle of triangle ABC. Which is tangent to AB, AC at F, E respectivly. Point D is the reflection of A through I_AB. Lines DI_A and EF meet at K. Prove that ,circumcenter of DKE , midpoint of BC and I_A are collinear.

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4 Triangle ABC is given. Let O be it's circumcenter. Let I be the center of it's incircle.The external angle bisector of A meet BC at D. And I_A is the A-excenter . The point K is chosen on the line AI such that $AK = 2AI$ and A is closer to K than I. If the segment DF is the diameter of the circumcircle of triangle DKI_A, then prove $OF = 3OI$.

– Combinatorics

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1 $1)$. Prove a graph with $2n$ vertices and $n + 2$ edges has an independent set of size n (there are n vertices such that no two of them are adjacent). $2)$.Find the number of graphs with $2n$ vertices and $n + 3$ edges , such that among any n vertices there is an edge connecting two of them

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2 For each n find the number of ways one can put the numbers $\{1, 2, 3, ..., n\}$ numbers on the circle, such that if for any 4 numbers a, b, c, d where $n|a + b - c - d$. The segments joining a, b and c, d do not meet inside the circle. (Two ways are said to be identical , if one can be obtained from rotaiting the other)

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3 Consider a latin square of size n. We are allowed to choose a 1×1 square in the table, and add 1 to any number on the same row and column as the chosen square (the original square will be counted aswell) , or we can add -1 to all of them instead. Can we with doing finitly many operation , reach any latin square of size n?

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4 What is the maximum number of subsets of size 5, taken from the set $A = \{1, 2, 3, ..., 20\}$ such that any 2 of them share exactly 1 element.

– Number Theory

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1 Find all positive integers n such that the following holds.

$$\tau(n)|2^{\sigma(n)} - 1$$

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2 Find all polynomials P with integer coefficients such that all the roots of $P^n(x)$ are integers. (here $P^n(x)$ means $P(P(...(P(x))...))$ where P is repeated n times)

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3 Find all functions f from positive integers to themselves, such that the followings hold. $1)$.for each positive integer n we have $f(n) < f(n + 1) < f(n) + 2020$. $2)$.for each positive integer n we have $S(f(n)) = f(S(n))$ where $S(n)$ is the sum of digits of n in base 10 representation.

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4 Prove that for every two positive integers a, b greater than 1. there exists infinitly many n such that the equation $\phi(a^n - 1) = b^m - b^t$ can't hold for any positive integers m, t.