Korea National Olympiad 2020

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1 Determine all functions $f:\mathbb{R}\rightarrow\mathbb{R}$ such that

$$x^2f(x)+yf(y^2)=f(x+y)f(x^2-xy+y^2)$$

for all $x,y\in\mathbb{R}$.

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2 H is the orthocenter of an acute triangle ABC, and let M be the midpoint of BC. Suppose (AH) meets AB and AC at D,E respectively. AH meets DE at P, and the line through H perpendicular to AH meets DM at Q. Prove that P,Q,B are collinear.

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3 There are n boys and m girls at Daehan Mathematical High School.
Let $d(B)$ a number of girls who know Boy B each other, and let $d(G)$ a number of boys who know Girl G each other.
Each girl knows at least one boy each other.
Prove that there exist Boy B and Girl G who knows each other in condition that $\frac{d(B)}{d(G)}\ge\frac{m}{n}$.

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4 Find a pair of coprime positive integers (m,n) other than $(41,12)$ such that m^2-5n^2 and m^2+5n^2 are both perfect squares.

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5 For some positive integer n, there exists n different positive integers $a_1, a_2, ..., a_n$ such that (1) $a_1=1, a_n=2000$ (2) $\forall i\in \mathbb{Z}$ $s.t.$ $2\le i\le n, a_i-a_{i-1}\in \{-3,5\}$
Determine the maximum value of n.

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6 Let $ABCDE$ be a convex pentagon such that quadrilateral $ABDE$ is a parallelogram and quadrilateral $BCDE$ is inscribed in a circle. The circle with center C and radius CD intersects the line BD, DE at points $F, G(\neq D)$, and points A, F, G is on line l. Let H be the intersection point of line l and segment BC.
Consider the set of circle Ω satisfying the following condition.

Circle Ω passes through A, H and intersects the sides AB, AE at point other than A.

Let $P, Q(\neq A)$ be the intersection point of circle Ω and sides AB, AE.
Prove that $AP+AQ$ is constant.
