Dutch IMO Team Selection Test 2020

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by parmenides51

–	Day 1	
1	In acute-angled triangle ABC, I is the center of the inscribed circle and holds $\|AC\| + \|AI\| = \|BC\|$. Prove that $\angle BAC = 2\angle ABC$.	
2	Determine all polynomials $P(x)$ with real coefficients that apply $P(x^2) + 2P(x) = P(x)^2 + 2$.	
3	For a positive integer n, we consider an $n \times n$ board and tiles with dimensions $1 \times 1, 1 \times 2, ..., 1 \times n$. In how many ways exactly can $\frac{1}{2}n(n+1)$ cells of the board are colored red, so that the red squares can all be covered by placing the n tiles all horizontally, but also by placing all n tiles vertically? Two colorings that are not identical, but by rotation or reflection from the board into each other count as different.	
4	Let $a, b \geq 2$ be positive integers with $gcd(a, b) = 1$. Let r be the smallest positive value that $\frac{a}{b} - \frac{c}{d}$ can take, where c and d are positive integers satisfying $c \leq a$ and $d \leq b$. Prove that $\frac{1}{r}$ is an integer.	
–	Day 2	
1	Given are real numbers $a_1, a_2, ..., a_{2020}$, not necessarily different. For every $n \geq 2020$, define a_{n+1} as the smallest real zero of the polynomial $$P_n(x) = x^{2n} + a_1x^{2n-2} + a_2x^{2n-4} + ... + a_{n-1}x^2 + a_n$$, if it exists. Assume that a_{n+1} exists for all $n \geq 2020$. Prove that $a_{n+1} \leq a_n$ for all $n \geq 2021$.	
2	Ward and Gabrielle are playing a game on a large sheet of paper. At the start of the game, there are 999 ones on the sheet of paper. Ward and Gabrielle each take turns alternatingly, and Ward has the first turn. During their turn, a player must pick two numbers a and b on the sheet such that $gcd(a, b) = 1$, erase these numbers from the sheet, and write the number $a + b$ on the sheet. The first player who is not able to do so, loses. Determine which player can always win this game.	
3	Find all pairs (a, b) of positive integers for which $a + b = \phi(a) + \phi(b) + gcd(a, b)$. Here $\phi(n)$ is the number of numbers k from $\{1, 2, ..., n\}$ with $gcd(n, k) = 1$.	

4 Let ABC be an acute-angled triangle and let P be the intersection of the tangents at B and C of the circumscribed circle of $\triangle ABC$. The line through A perpendicular on AB and cuts the line perpendicular on AC through C at X. The line through A perpendicular on AC cuts the line perpendicular on AB through B at Y. Show that $AP \perp XY$.

– Day 3

1 For a positive number n, we write $d(n)$ for the number of positive divisors of n.
Determine all positive integers k for which exist positive integers a and b with the property $k = d(a) = d(b) = d(2a + 3b)$.

2 Given is a triangle ABC with its circumscribed circle and $|AC| < |AB|$. On the short arc AC, there is a variable point $D \neq A$. Let E be the reflection of A wrt the inner bisector of $\angle BDC$. Prove that the line DE passes through a fixed point, regardless of point D.

3 Find all functions $f : Z \to Z$ that satisfy

$$f(-f(x) - f(y)) = 1 - x - y$$

for all $x, y \in Z$

4 Given are two positive integers k and n with $k \le n \le 2k - 1$. Julian has a large stack of rectangular $k \times 1$ tiles. Merlin calls a positive integer m and receives m tiles from Julian to place on an $n \times n$ board. Julian first writes on every tile whether it should be a horizontal or a vertical tile. Tiles may be used the board should not overlap or protrude. What is the largest number m that Merlin can call if he wants to make sure that he has all tiles according to the rule of Julian can put on the plate?