Austrian Federal Competition For Advanced Students, P2, 2020

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–	Day 1
1	Let $ABCD$ be a convex cyclic quadrilateral with the diagonal intersection S. Let further be P the circumcenter of the triangle ABS and Q the circumcenter of the triangle BCS. The parallel to AD through P and the parallel to CD through Q intersect at point R. Prove that R is on BD. (Karl Czakler)
2	In the plane there are 2020 points, some of which are black and the rest are green. For every black point, the following applies: [i]There are exactly two green points that represent the distance 2020 from that black point. [/i] Find the smallest possible number of green dots. (Walther Janous)
3	Let a be a fixed positive integer and (e_n) the sequence, which is defined by $e_0 = 1$ and $$e_n = a + \prod_{k=0}^{n-1} e_k$$ for $n \geq 1$. Prove that (a) There exist infinitely many prime numbers that divide one element of the sequence. (b) There exists one prime number that does not divide an element of the sequence. (Theresia Eisenkölbl)
–	Day 2
4	Determine all functions $f : \mathbb{R} \to \mathbb{R}$, such that $$f(xf(y) + 1) = y + f(f(x)f(y))$$ for all $x, y \in \mathbb{R}$. (Theresia Eisenkölbl)
5	Let h be a semicircle with diameter AB. Let P be an arbitrary point inside the diameter AB. The perpendicular through P on AB intersects h at point C. The line PC divides the semicircular

area into two parts. A circle will be inscribed in each of them that touches AB, PC and h. The points of contact of the two circles with AB are denoted by D and E, where D lies between A and P. Prove that the size of the angle DCE does not depend on the choice of P.

(Walther Janous)

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6 The players Alfred and Bertrand put together a polynomial $x^n + a_{n-1}x^{n-1} + ... + a_0$ with the given degree $n \geq 2$. To do this, they alternately choose the value in n moves one coefficient each, whereby all coefficients must be integers and $a_0 \neq 0$ must apply. Alfred's starts first . Alfred wins if the polynomial has an integer zero at the end.
(a) For which n can Alfred force victory if the coefficients a_j are from the right to the left, i.e. for $j = 0, 1, ..., n-1$, be determined?
(b) For which n can Alfred force victory if the coefficients a_j are from the left to the right, i.e. for $j = n-1, n-2, ..., 0$, be determined?

(Theresia Eisenklbl, Clemens Heuberger)
