**Dutch Mathematical Olympiad 2020**

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1   Daan distributes the numbers 1 to 9 over the nine squares of a 3×3-table (each square receives exactly one number). Then, in each row, Daan circles the median number (the number that is neither the smallest nor the largest of the three). For example, if the numbers $8, 1$, and 2 are in one row, he circles the number 2. He does the same for each column and each of the two diagonals. If a number is already circled, he does not circle it again. He calls the result of this process a *median table*. Above, you can see a median table that has 5 circled numbers.
(a) What is the **smallest** possible number of circled numbers in a median table?
Prove that a smaller number is not possible and give an example in which a minimum number of numbers is circled.
(b) What is the **largest** possible number of circled numbers in a median table?
Prove that a larger number is not possible and give an example in which a maximum number of numbers is circled.

2   For a given value t, we consider number sequences $a_1, a_2, a_3, \ldots$ such that $a_{n+1} = \frac{a_n + t}{a_n + 1}$ for all $n \geq 1$.

(a) Suppose that $t = 2$. Determine all starting values $a_1 > 0$ such that $\frac{4}{3} \leq a_n \leq \frac{3}{2}$ holds for all $n \geq 2$.

(b) Suppose that $t = -3$. Investigate whether $a_{2020} = a_1$ for all starting values a_1 different from -1 and 1.

3   Given is a parallelogram $ABCD$ with $\angle A < 90^o$ and $|AB| < |BC|$. The angular bisector of angle A intersects side BC in M and intersects the extension of DC in N. Point O is the centre of the circle through M, C, and N. Prove that $\angle OBC = \angle ODC$.
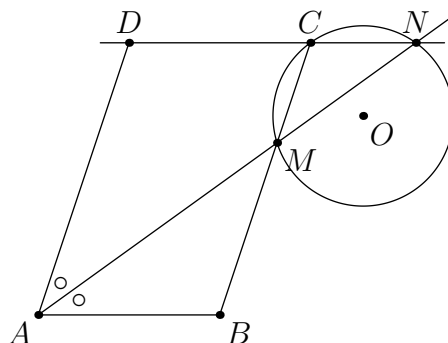
4   Determine all pairs of integers (x, y) such that $2xy$ is a perfect square and $x^2 + y^2$ is a prime number.

5   Sabine has a very large collection of shells. She decides to give part of her collection to her sister.

On the first day, she lines up all her shells. She takes the shells that are in a position that is a perfect square (the first, fourth, ninth, sixteenth, etc. shell), and gives them to her sister. On the second day, she lines up her remaining shells. Again, she takes the shells that are in a position that is a perfect square, and gives them to her sister. She repeats this process every day.

The 27th day is the first day that she ends up with fewer than 1000 shells. The 28th day she ends up with a number of shells that is a perfect square for the tenth time.

What are the possible numbers of shells that Sabine could have had in the very beginning?