Czech And Slovak Mathematical Olympiad, Round III, Category A 2020

www.artofproblemsolving.com/community/c1617611

by parmenides51

1 Two positive integers m and n are written on the board.
We replace one of two numbers in each step on the board by either their sum, or product, or ratio (if it is an integer).
Depending on the numbers m and n, specify all the pairs that can appear on the board in pairs.

(Radovan varc)

---

2 The triangle ABC is given. Inside its sides AB and AC, the points X and Y are respectively selected Let Z be the intersection of the lines BY and CX. Prove the inequality

$$[BZX] + [CZY] > 2[XYZ]$$

, where $[DEF]$ denotes the content of the triangle DEF.

(David Hruska, Josef Tkadlec)

---

3 Consider the system of equations $\begin{cases} x^2 - 3y + p = z, \\ y^2 - 3z + p = x, \\ z^2 - 3x + p = y \end{cases}$ with real parameter p.
a) For $p \geq 4$, solve the considered system in the field of real numbers.
b) Prove that for $p \in (1, 4)$ every real solution of the system satisfies $x = y = z$.

(Jaroslav Svrcek)

---

4 Positive integers a, b satisfy equality $b^2 = a^2 + ab + b$.
Prove that b is a square of a positive integer.

(Patrik Bak)

---

5 Given an isosceles triangle ABC with base BC. Inside the side BC is given a point D. Let E, F be respectively points on the sides AB, AC that $|\angle BED| = |\angle DFC| > 90^o$. Prove that the circles circumscribed around the triangles ABF and AEC intersect on the line AD at a point different from point A.

(Patrik Bak, Michal Rolnek)

---

6 For each positive integer k, denote by $P(k)$ the number of all positive integers $4k$-digit numbers which can be composed of the digits $2, 0$ and which are divisible by the number 2020. Prove the inequality

$$P(k) \geq \binom{2k-1}{k}^2$$

and determine all k for which equality occurs.

(Note: A positive integer cannot begin with a digit of 0.)

(Jaromir Simsa)