2021 China National Olympiad

www.artofproblemsolving.com/community/c1617667

by Tintarn, Rickyminer, mofumofu, sqing, SpecialBeing2017, NTssu

– Day 1

1 Let $\{z_n\}_{n\ge 1}$ be a sequence of complex numbers, whose odd terms are real, even terms are purely imaginary, and for every positive integer k, $|z_k z_{k+1}|=2^k$. Denote $f_n=|z_1+z_2+\cdots+z_n|$, for $n=1,2,\cdots$
(1) Find the minimum of f_{2020}.
(2) Find the minimum of $f_{2020}\cdot f_{2021}$.

2 Let $m>1$ be an integer. Find the smallest positive integer n, such that for any integers $a_1,a_2,\ldots,a_n; b_1,b_2,\ldots,b_n$ there exists integers $x_1,x_2,\ldots,x_n$ satisfying the following two conditions:

i) There exists $i\in\{1,2,\ldots,n\}$ such that x_i and m are coprime

ii) $\sum_{i=1}^n a_ix_i \equiv \sum_{i=1}^n b_ix_i \equiv 0 \pmod m$

3 Let n be positive integer such that there are exactly 36 different prime numbers that divides n. For $k=1,2,3,4,5$, c_n be the number of integers that are mutually prime numbers to n in the interval $[\frac{(k-1)n}{5},\frac{kn}{5}]$. c_1,c_2,c_3,c_4,c_5 is not exactly the same.Prove that

$$\sum_{1\le i<j\le 5}(c_i-c_j)^2\ge 2^{36}.$$

– Day 2

4 In acute triangle $ABC(AB>AC)$, M is the midpoint of minor arc BC, O is the circumcenter of (ABC) and AK is its diameter. The line parallel to AM through O meets segment AB at D, and CA extended at E. Lines BM and CK meet at P, lines BK and CM meet at Q. Prove that $\angle OPB+\angle OEB=\angle OQC+\angle ODC$.

5 P is a convex polyhedron such that:

(1) every vertex belongs to exactly 3 faces.

(1) For every natural number n, there are even number of faces with n vertices.

An ant walks along the edges of P and forms a non-self-intersecting cycle, which divides the faces of this polyhedron into two sides, such that for every natural number n, the number of faces with n vertices on each side are the same. (assume this is possible)

Show that the number of times the ant turns left is the same as the number of times the ant turn right.

---

6 Find $f : \mathbb{Z}_+ \to \mathbb{Z}_+$, such that for any $x, y \in \mathbb{Z}_+$,

$$f(f(x) + y) \mid x + f(y).$$
