Northern Mathematical Olympiad 2005

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–	Day 1
1	AB is a chord of a circle with center O, M is the midpoint of AB. A non-diameter chord is drawn through M and intersects the circle at C and D. The tangents of the circle from points C and D intersect line AB at P and Q, respectively. Prove that PA = QB.
2	Let f be a function from R to R. Suppose we have: (1) $f(0)=0$ (2) For all $x, y \in (-\infty, -1) \cup (1, \infty)$, we have $f(\frac{1}{x}) + f(\frac{1}{y}) = f(\frac{x+y}{1+xy})$. (3) If $x \in (-1, 0)$, then $f(x) > 0$. Prove: $\sum_{n=1}^{+\infty} f(\frac{1}{n^2+7n+11}) > f(\frac{1}{2})$ with $n \in N^+$.
3	Let positive numbers $a_1, a_2, ..., a_{3n}$ $(n \geq 2)$ constitute an arithmetic progression with common difference $d > 0$. Prove that among any $n+2$ terms in this progression, there exist two terms a_i, a_j $(i \neq j)$ satisfying $1 < \frac{\lvert a_i - a_j \rvert}{nd} < 2$.
–	Day 2
4	Let A be the set of n-digit integers whose digits are all from $\{1,2,3,4,5\}$. B is subset of A such that it contains digit 5, and there is no digit 3 in front of digit 5 (i.e. for $n=2$, 35 is not allowed, but 53 is allowed). How many elements does set B have?
5	Let x, y, z be positive real numbers such that $x^2+xy+y^2 = \frac{25}{4}$, $y^2+yz+z^2 = 36$, and $z^2+zx+x^2 = \frac{169}{4}$. Find the value of $xy+yz+zx$.
6	Let $0 \leq \alpha, \beta, \gamma \leq \frac{\pi}{2}$, such that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$. Prove that $2 \leq (1+\cos^2\alpha)^2\sin^4\alpha + (1+\cos^2\beta)^2\sin^4\beta + (1+\cos^2\gamma)^2\sin^4\gamma \leq (1+\cos^2\alpha)(1+\cos^2\beta)(1+\cos^2\gamma)$.