Northern Mathematical Olympiad 2006

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–	Day 1
1	AB is the diameter of circle O, CD is a non-diameter chord that is perpendicular to AB. Let E be the midpoint of OC, connect AE and extend it to meet the circle at point P. Let DP and BC meet at F. Prove that F is the midpoint of BC.
2	p is a prime number that is greater than 2. Let $\{a_n\}$ be a sequence such that $na_{n+1} = (n+1)a_n - \left(\frac{p}{2}\right)^4$. Show that if $a_1 = 5$, the $16 \mid a_{81}$.
3	AD is the altitude on side BC of triangle ABC. If $BC + AD - AB - AC = 0$, find the range of $\angle BAC$. *Alternative formulation.* Let AD be the altitude of triangle ABC to the side BC. If $BC + AD = AB + AC$, then find the range of $\angle A$.
4	Given a function $f(x) = x^2 + ax + b$ with $a, b \in R$, if there exists a real number m such that $\lvert f(m)\rvert \le \frac{1}{4}$ and $\lvert f(m+1)\rvert \le \frac{1}{4}$, then find the maximum and minimum of the value of $\Delta = a^2 - 4b$.
–	Day 2
5	a, b, c are positive numbers such that $a + b + c = 3$, show that: $$\frac{a^2+9}{2a^2+(b+c)^2} + \frac{b^2+9}{2b^2+(a+c)^2} + \frac{c^2+9}{2c^2+(a+b)^2} \le 5$$
6	canceled
7	Can we put positive integers $1, 2, 3, \cdots 64$ into 8×8 grids such that the sum of the numbers in any 4 grids that have the form like T (3 on top and 1 under the middle one on the top, this can be rotate to any direction) can be divided by 5?
8	Given a sequence $\{a_n\}$ such that $a_{n+1} = a_n + \frac{1}{2006}a_n^2$, $n \in N$, $a_0 = \frac{1}{2}$. Prove that $1 - \frac{1}{2008} < a_{2006} < 1$.

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