European Mathematical Cup 2020

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by parmenides51, square_root_of_3, Adilet160205

–	Junior Division
1	Let ABC be an acute-angled triangle. Let D and E be the midpoints of sides $\overline{AB}$ and $\overline{AC}$ respectively. Let F be the point such that D is the midpoint of $\overline{EF}$. Let Γ be the circumcircle of triangle FDB. Let G be a point on the segment $\overline{CD}$ such that the midpoint of $\overline{BG}$ lies on Γ. Let H be the second intersection of Γ and FC. Show that the quadrilateral $BHGC$ is cyclic. *Proposed by Art Waeterschoot.*
2	A positive integer $k \geqslant 3$ is called *fibby* if there exists a positive integer n and positive integers $d_1 < d_2 < \ldots < d_k$ with the following properties: • $d_{j+2} = d_{j+1} + d_j$ for every j satisfying $1 \leqslant j \leqslant k-2$, • $d_1, d_2, \ldots, d_k$ are divisors of n, • any other divisor of n is either less than d_1 or greater than d_k. Find all fibby numbers. *Proposed by Ivan Novak.*
3	Two types of tiles, depicted on the figure below, are given. `https://wiki-images.artofproblemsolving.com//2/23/Izrezak.PNG` Find all positive integers n such that an $n \times n$ board consisting of n^2 unit squares can be covered without gaps with these two types of tiles (rotations and reflections are allowed) so that no two tiles overlap and no part of any tile covers an area outside the $n \times n$ board. *Proposed by Art Waeterschoot*
4	Let a, b, c be positive real numbers such that $ab+bc+ac = a+b+c$. Prove the following inequality: $$\sqrt{a+\frac{b}{c}}+\sqrt{b+\frac{c}{a}}+\sqrt{c+\frac{a}{b}} \leq \sqrt{2}\cdot\min\left\{\frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ \frac{b}{a}+\frac{c}{b}+\frac{a}{c}\right\}.$$ *Proposed by Dorlir Ahmeti.*
–	Senior Division

1 Let $ABCD$ be a parallelogram such that $|AB| > |BC|$. Let O be a point on the line CD such that $|OB| = |OD|$. Let ω be a circle with center O and radius $|OC|$. If T is the second intersection of ω and CD, prove that AT, BO and ω are concurrent.

Proposed by Ivan Novak

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2 Let n and k be positive integers. An n-tuple $(a_1, a_2, \ldots, a_n)$ is called a permutation if every number from the set $\{1, 2, ..., n\}$ occurs in it exactly once. For a permutation $(p_1, p_2, ..., p_n)$, we define its k-mutation to be the n-tuple

$$(p_1 + p_{1+k}, p_2 + p_{2+k}, ..., p_n + p_{n+k}),$$

where indices are taken modulo n. Find all pairs (n, k) such that every two distinct permutations have distinct k-mutations.

Remark: For example, when $(n, k) = (4, 2)$, the 2-mutation of $(1, 2, 4, 3)$ is $(1+4, 2+3, 4+1, 3+2) = (5, 5, 5, 5)$.

Proposed by Borna Šimić

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3 Let p be a prime number. Troy and Abed are playing a game. Troy writes a positive integer X on the board, and gives a sequence $(a_n)_{n\in\mathbb{N}}$ of positive integers to Abed. Abed now makes a sequence of moves. The n-th move is the following:

Replace Y currently written on the board with either $Y + a_n$ or $Y \cdot a_n$.

Abed wins if at some point the number on the board is a multiple of p. Determine whether Abed can win, regardless of Troy's choices, if $a)p = 10^9 + 7$; $b)p = 10^9 + 9$.
Remark: Both $10^9 + 7$ and $10^9 + 9$ are prime.

Proposed by Ivan Novak

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4 Let $\mathbb{R}^+$ denote the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$xf(x + y) + f(xf(y) + 1) = f(xf(x))$$

for all $x, y \in \mathbb{R}^+$.

Proposed by Amadej Kristjan Kocbek, Jakob Jurij Snoj
