VMO 2021

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– Day 1

1 Let (x_n) define by $x_1 \in \left(0; \frac{1}{2}\right)$ and $x_{n+1} = 3x_n^2 - 2nx_n^3$ for all $n \geq 1$.

a) Prove that (x_n) convergence to 0.

b) For each $n \geq 1$, let $y_n = x_1 + 2x_2 + \cdots + nx_n$. Prove that (y_n) has a limit.

2 Find all function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x)f(y) = f(xy - 1) + yf(x) + xf(y)$$

for all $x, y \in \mathbb{R}$

3 Let $\triangle ABC$ is not an isosceles triangle and is an acute triangle, AD, BE, CF be the altitudes and H is the orthocenter .Let I is the circumcenter of $\triangle HEF$ and let K, J is the midpoint of BC, EF respectively.Let HJ intersects (I) again at G and GK intersects (I) at $L \neq G$.
a) Prove that AL is perpendicular to EF.
b) Let AL intersects EF at M, the line IM intersects the circumcircle $\triangle IEF$ again at N, DN intersects AB, AC at P and Q respectively then prove that PE, QF, AK are concurrent.

4 For an integer $n \geq 2$, let $s(n)$ be the sum of positive integers not exceeding n and not relatively prime to n.
a) Prove that $s(n) = \frac{n}{2}(n + 1 - \varphi(n))$, where $\varphi(n)$ is the number of integers positive cannot exceed n and are relatively prime to n.
b) Prove that there is no integer $n \geq 2$ such that $s(n) = s(n + 2021)$

– Day 2

5 Let the polynomial $P(x) = a_{21}x^{21} + a_{20}x^{20} + \cdots + a_1x + a_0$ where $1011 \leq a_i \leq 2021$ for all $i = 0, 1, 2, ..., 21$. Given that $P(x)$ has an integer root and there exists an positive real number$c$ such that $|a_{k+2} - a_k| \leq c$ for all $k = 0, 1, ..., 19$.

a) Prove that $P(x)$ has an only integer root.

b) Prove that

$$\sum_{k=0}^{10} (a_{2k+1} - a_{2k})^2 \leq 440c^2.$$

6 A student divides all 30 marbles into 5 boxes numbered $1, 2, 3, 4, 5$ (after being divided, there may be a box with no marbles).
a) How many ways are there to divide marbles into boxes (are two different ways if there is a box with a different number of marbles)?
b) After dividing, the student paints those 30 marbles by a number of colors (each with the same color, one color can be painted for many marbles), so that there are no 2 marbles in the same box. have the same color and from any 2 boxes it is impossible to choose 8 marbles painted in 4 colors. Prove that for every division, the student must use no less than 10 colors to paint the marbles.
c) Show a division so that with exactly 10 colors the student can paint the marbles that satisfy the conditions in question b).

7 Let ABC be an inscribed triangle in circle (O). Let D be the intersection of the two tangent lines of (O) at B and C. The circle passing through A and tangent to BC at B intersects the median passing A of the triangle ABC at G. Lines BG, CG intersect CD, BD at E, F respectively.
a) The line passing through the midpoint of BE and CF cuts BF, CE at M, N respectively. Prove that the points A, D, M, N belong to the same circle.
b) Let AD, AG intersect the circumcircle of the triangles DBC, GBC at H, K respectively. The perpendicular bisectors of HK, HE, and HF cut BC, CA, and AB at R, P, and Q respectively. Prove that the points R, P, and Q are collinear.