**Problems from 2020 Thailand Mathematical Olympiad**

www.artofproblemsolving.com/community/c1678538

by MarkBcc168, Mathmick51

**–** Day 1

**1** Show that $\varphi(2n) \mid n!$ for all positive integer $n$.

**2** There are $63$ houses at the distance of $1, 2, 3, ..., 63$ km from the north pole, respectively. Santa Clause wants to distribute vaccine to each house. To do so, he will let his assistants, $63$ elfs named $E_1, E_2, ..., E_{63}$ , deliever the vaccine to each house; each elf will deliever vaccine to exactly one house and never return. Suppose that the elf $E_n$ takes $n$ minutes to travel $1$ km for each $n = 1, 2, ..., 63$ , and that all elfs leave the north pole simultaneously. What is the minimum amount of time to complete the delivery?

**3** Suppose that $f : \mathbb{R}^+ \to \mathbb{R}$ satisfies the equation

$$f(a+b+c+d) = f(a) + f(b) + f(c) + f(d)$$

for all $a, b, c, d$ that are the four sides of some tangential quadrilateral. Show that $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^+$.

**4** Let $\triangle ABC$ be a triangle with altitudes $AD, BE, CF$. Let the lines $AD$ and $EF$ meet at $P$, let the tangent to the circumcircle of $\triangle ADC$ at $D$ meet the line $AB$ at $X$, and let the tangent to the circumcircle of $\triangle ADB$ at $D$ meet the line $AC$ at $Y$. Prove that the line $XY$ passes through the midpoint of $DP$.

**5** You have an $n \times n$ grid and want to remove all edges of the grid by the sequence of the following moves. In each move, you can select a cell and remove exactly three edges surrounding that cell; in particular, that cell must have at least three remaining edges for the operation to be valid. For which positive integers $n$ is this possible?

**–** Day 2

**6** Let the incircle of an acute triangle $\triangle ABC$ touches $BC, CA$, and $AB$ at points $D, E$, and $F$, respectively. Place point $K$ on the side $AB$ so that $DF$ bisects $\angle ADK$, and place point $L$ on the side $AB$ so that $EF$ bisects $\angle BEL$.

-Prove that $\triangle ALE \sim \triangle AEB$.
-Prove that $FK = FL$.

**7** Determine all functions $f : \mathbb{R} \to \mathbb{Z}$ satisfying the inequality $(f(x))^2 + (f(y))^2 \leq 2f(xy)$ for all reals $x, y$.

**8** For all positive real numbers $a, b, c$ with $a+b+c=3$, prove the inequality

$$\frac{a^6}{c^2+2b^3}+\frac{b^6}{a^2+2c^3}+\frac{c^6}{b^2+2a^3} \geq 1.$$

**9** Let $n, k$ be positive integers such that $n > k$. There is a square-shaped plot of land, which is divided into $n \times n$ grid so that each cell has the same size. The land needs to be plowed by $k$ tractors; each tractor will begin on the lower-left corner cell and keep moving to the cell sharing a common side until it reaches the upper-right corner cell. In addition, each tractor can only move in two directions: up and right. Determine the minimum possible number of unplowed cells.

**10** Determine all polynomials $P(x)$ with integer coefficients which satisfies $P(n) \mid n!+2$ for all postive integer $n$.