Hong Kong Team Selection Test 2016

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Test 1 August 2015

1 Find all natural numbers n such that n, n^2+10, n^2-2, n^3+6, and n^5+36 are all prime numbers.

2 Find the largest possible positive integer n , so that there exist$n$ distinct positive real numbers $x_1, x_2, ..., x_n$ satisfying the following inequality : for any $1 \leq i, j \leq n$, $(3x_i - x_j)(x_i - 3x_j) \geq (1 - x_i x_j)^2$

3 Let ABC be a triangle such that $AB \neq AC$. The incircle with centre I touches BC at D. Line AI intersects the circumcircle Γ of ABC at M, and DM again meets Γ at P. Find $\angle API$

4 Find all triples (m, p, q) such that

$$2^m p^2 + 1 = q^7,$$

where p and q are ptimes and m is a positive integer.

5 Let $ABCD$ be inscribed in a circle with center O. Let E be the intersection of AC and BD. M and N are the midpoints of the arcs AB and CD respectively (the arcs not containing any other vertices). Let P be the intersection point of EO and MN. Suppose $BC = 5$, $AC = 11$, $BD = 12$, and $AD = 10$. Find $\frac{MN}{NP}$

6 4031 lines are drawn on a plane, no two parallel or perpendicular, and no three lines meet at a point. Determine the maximum number of acute-angled triangles that may be formed.

Test 2 October 2016

1 During a school year 44 competitions were held. Exactly 7 students won in each of the competition. For any two competitions, there exists exactly 1 student who won both competitions. Is it true that there exists a student who won all the competitions?

2 Let Γ be a circle and AB be a diameter. Let l be a line outside the circle, and is perpendicular to AB. Let X, Y be two points on l. If X', Y' are two points on l such that AX, BX' intersect on Γ and such that AY, BY' intersect on Γ. Prove that the circumcircles of triangles AXY and $AX'Y'$ intersect at a point on Γ other than A, or the three circles are tangent at A.

3 Let a, b, c be positive real numbers satisfying $abc = 1$. Determine the smallest possible value of

$$\frac{a^3+8}{a^3(b+c)} + \frac{b^3+8}{b^3(a+c)} + \frac{c^3+8}{c^3(b+a)}$$

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4 Mable and Nora play a game according to the following steps in order:

1. Mable writes down any 2015 distinct prime numbers in ascending order in a row. The product of these primes is Marble's score.

2. Nora writes down a positive integer

3. Mable draws a vertical line between two adjacent primes she has written in step 1, and compute the product of the prime(s) on the left of the vertical line

4. Nora must add the product obtained by Marble in step 3 to the number she has written in step 2, and the sum becomes Nora's score.

If Marble and Nora's scores have a common factor greater than 1, Marble wins, otherwise Nora wins.

Who has a winning strategy?

Note There are 2 more tests between test 2 and 3 that are accounted for in the IMO selection process, namely the CHKMO and the APMO.

Test 3 24 April 2016

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1 Find all prime numbers p and q such that $p^2|q^3+1$ and $q^2|p^6-1$

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2 Suppose that I is the incenter of triangle ABC. The perpendicular to line AI from point I intersects sides AC and AB at points B' and C' respectively. Points B_1 and C_1 are placed on half lines BC and CB respectively, in such a way that $AB = BB_1$ and $AC = CC_1$. If T is the second intersection point of the circumcircles of triangles AB_1C' and AC_1B', prove that the circumcenter of triangle ATI lies on the line BC

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3 2016 circles with radius 1 are lying on the plane. Among these 2016 circles, show that one can select a collection C of 27 circles satisfying the following: either every pair of two circles in C intersect or every pair of two circles in C does not intersect.

Test 4 29 April 2016

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1 Let O be the circumcenter of a triangle ABC, and let l be the line going through the midpoint of the side BC and is perpendicular to the bisector of $\angle BAC$. Determine the value of $\angle BAC$ if the line l goes through the midpoint of the line segment AO.

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2 Determine all positive integers n for which there exist pairwise distinct positive real numbers $a_1, a_2, \cdots, a_n$ satisfying $\left\{a_i + \dfrac{(-1)^i}{a_i} \mid 1 \leq i \leq n\right\} = \{a_i \mid 1 \leq i \leq n\}$

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3 Let p be a prime number greater than 5. Suppose there is an integer k satisfying that k^2+5 is divisible by p. Prove that there are positive integers m and n such that $p^2 = m^2 + 5n^2$
