Brazil National Olympiad 2015

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– Day 1

1 Let $\triangle ABC$ be an acute-scalene triangle, and let N be the center of the circle wich pass trough the feet of altitudes. Let D be the intersection of tangents to the circumcircle of $\triangle ABC$ at B and C. Prove that A, D and N are collinear iff $\measuredangle BAC = 45$.

2 Consider $S = \{1, 2, 3, \cdots, 6n\}$, $n > 1$. Find the largest k such that the following statement is true: every subset A of S with $4n$ elements has at least k pairs (a, b), $a < b$ and b is divisible by a.

3 Given a natural $n > 1$ and its prime fatorization $n = p_1^{\alpha 1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, its *false derived* is defined by

$$f(n) = \alpha_1 p_1^{\alpha_1 - 1} \alpha_2 p_2^{\alpha_2 - 1} ... \alpha_k p_k^{\alpha_k - 1}.$$

Prove that there exist infinitely many naturals n such that $f(n) = f(n-1) + 1$.

– Day 2

4 Let n be a integer and let $n = d_1 > d_2 > \cdots > d_k = 1$ its positive divisors.
a) Prove that

$$d_1 - d_2 + d_3 - \cdots + (-1)^{k-1} d_k = n - 1$$

iff n is prime or $n = 4$.
b) Determine the three positive integers such that

$$d_1 - d_2 + d_3 - ... + (-1)^{k-1} d_k = n - 4.$$

5 Is that true that there exist a polynomial $f(x)$ with rational coefficients, not all integers, with degree $n > 0$, a polynomial $g(x)$, with integer coefficients, and a set S with $n+1$ integers such that $f(t) = g(t)$ for all $t \in S$?

6 Let $\triangle ABC$ be a scalene triangle and X, Y and Z be points on the lines BC, AC and AB, respectively, such that $\measuredangle AXB = \measuredangle BYC = \measuredangle CZA$. The circumcircles of BXZ and CXY intersect at P. Prove that P is on the circumference which diameter has ends in the ortocenter H and in the baricenter G of $\triangle ABC$.