Taiwan National Olympiad 2001

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– Day 1

1 Let A be a set with at least 3 integers, and let M be the maximum element in A and m the minimum element in A. it is known that there exist a polynomial P such that: $m < P(a) < M$ for all a in A. And also $p(m) < p(a)$ for all a in $A - (m, M)$. Prove that $n < 6$ and there exist integers b and c such that $p(x) + x^2 + bx + c$ is cero in A.

2 Let $a_1, a_2, ..., a_{15}$ be positive integers for which the number $a_k^{k+1} - a_k$ is not divisible by 17 for any $k = 1, ..., 15$. Show that there are integers $b_1, b_2, ..., b_{15}$ such that:
(i) $b_m - b_n$ is not divisible by 17 for $1 \leq m < n \leq 15$, and
(ii) each b_i is a product of one or more terms of (a_i).

3 Let $n \geq 3$ be an integer and let $A_1, A_2, \ldots, A_n$ be n distinct subsets of $S = \{1, 2, \ldots, n\}$. Show that there exists $x \in S$ such that the n subsets $A_i - \{x\}, i = 1, 2, \ldots n$ are also disjoint.

what i have is we may assume that the union of the A_is is S.

– Day 2

4 Let Γ be the circumcircle of a fixed triangle ABC, and let M and N be the midpoints of the arcs BC and CA, respectively. For any point X on the arc AB, let O_1 and O_2 be the incenters of $\triangle XAC$ and $\triangle XBC$, and let the circumcircle of $\triangle XO_1O_2$ intersect Γ at X and Q. Prove that triangles QNO_1 and QMO_2 are similar, and find all possible locations of point Q.

5 Let $f(n) = \sum_{k=0}^{n-1} x^k y^{n-1-k}$ with, x, y real numbers. If $f(n)$, $f(n+1)$, $f(n+2)$, $f(n+3)$, are integers for some n, prove $f(n)$ is integer for all n.

6 Suppose that $n-1$ items $A_1, A_2, ..., A_{n-1}$ have already been arranged in the increasing order, and that another item A_n is to be inserted to preserve the order. What is the expected number of comparisons necessary to insert A_n?