International Zhautykov Olympiad 2021

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by Inshaallahgoldmedal, Adilet160205, CID-42, IndoMathXdZ

1 Prove that there exists a positive integer n, such that the remainder of 3^n when divided by 2^n is greater than 10^{2021}.

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2 In a convex cyclic hexagon $ABCDEF$, $BC = EF$ and $CD = AF$. Diagonals AC and BF intersect at point Q, and diagonals EC and DF intersect at point P. Points R and S are marked on the segments DF and BF respectively so that $FR = PD$ and $BQ = FS$. **The segments** RQ and PS intersect at point T. Prove that the line TC bisects the diagonal DB.

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3 Let $n \geq 2$ be an integer. Elwyn is given an $n \times n$ table filled with real numbers (each cell of the table contains exactly one number). We define a *rook set* as a set of n cells of the table situated in n distinct rows as well as in n distinct columns. Assume that, for every rook set, the sum of n numbers in the cells forming the set is nonnegative.

By a move, Elwyn chooses a row, a column, and a real number a, and then he adds a to each number in the chosen row, and subtracts a from each number in the chosen column (thus, the number at the intersection of the chosen row and column does not change). Prove that Elwyn can perform a sequence of moves so that all numbers in the table become nonnegative.

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4 Let there be an incircle of triangle ABC, and 3 circles each inscribed between incircle and angles of ABC.
Let r, r_1, r_2, r_3 be radii of these circles ($r_1, r_2, r_3 < r$). Prove that

$$r_1 + r_2 + r_3 \geq r$$

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5 On a party with 99 guests, hosts Ann and Bob play a game (the hosts are not regarded as guests). There are 99 chairs arranged in a circle; initially, all guests hang around those chairs. The hosts take turns alternately. By a turn, a host orders any standing guest to sit on an unoccupied chair c. If some chair adjacent to c is already occupied, the same host orders one guest on such chair to stand up (if both chairs adjacent to c are occupied, the host chooses exactly one of them). All orders are carried out immediately. Ann makes the first move; her goal is to fulfill, after some move of hers, that at least k chairs are occupied. Determine the largest k for which Ann can reach the goal, regardless of Bob's play.

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6 Let $P(x)$ be a nonconstant polynomial of degree n with rational coefficients which can not be presented as a product of two nonconstant polynomials with rational coefficients. Prove that the number of polynomials $Q(x)$ of degree less than n with rational coefficients such that $P(x)$ divides $P(Q(x))$

a) is finite

b) does not exceed n.