Argentine National Olympiad 2014, Level 3

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Day 1 12 November, 2014

1. 201 positive integers are written on a line, such that both the first one and the last one are equal to 19999. Each one of the remaining numbers is less than the average of its neighbouring numbers, and the differences between each one of the remaining numbers and the average of its neighbouring numbers are all equal to a unique integer. Find the second-to-last term on the line.

2. Given several numbers, one of them, a, is chosen and replaced by the three numbers $\frac{a}{3}, \frac{a}{3}, \frac{a}{3}$. This process is repeated with the new set of numbers, and so on. Originally, there are 1000 ones, and we apply the process several times. A number m is called *good* if there are m or more numbers that are the same after each iteration, no matter how many or what operations are performed. Find the largest possible good number.

3. Two circumferences of radius 1 that do not intersect, c_1 and c_2, are placed inside an angle whose vertex is O. c_1 is tangent to one of the rays of the angle, while c_2 is tangent to the other ray. One of the common internal tangents of c_1 and c_2 passes through O, and the other one intersects the rays of the angle at points A and B, with $AO = BO$. Find the distance of point A to the line OB.

Day 2 13 November, 2014

4. Consider the following 50-term sums:

$S = \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \ldots + \frac{1}{99\cdot 100}$,

$T = \frac{1}{51\cdot 100} + \frac{1}{52\cdot 99} + \ldots + \frac{1}{99\cdot 52} + \frac{1}{100\cdot 51}$.

Express $\frac{S}{T}$ as an irreducible fraction.

5. An integer $n \geq 3$ is called *special* if it does not divide $(n-1)!\left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right)$. Find all special numbers n such that $10 \leq n \leq 100$.

6. Determine whether there exists positive integers $a_1 < a_2 < \cdots < a_k$ such that all sums $a_i + a_j$, where $1 \leq i < j \leq k$, are unique, and among those sums, there are 1000 consecutive integers.