Western Mathematical Olympiad 2015

www.artofproblemsolving.com/community/c177736

by buzzychaoz, sqing, YIOwOIY

Day 1

1 Let the integer $n \geq 2$, and $x_1, x_2, \cdots, x_n$ be real numbers such that $\sum_{k=1}^{n} x_k$ be integer . $d_k = \min_{m \in Z} |x_k - m|$, $1 \leq k \leq n$ .Find the maximum value of $\sum_{k=1}^{n} d_k$.

2 Two circles (Ω_1) , (Ω_2) touch internally on the point T. Let M, N be two points on the circle (Ω_1) which are different from T and A, B, C, D be four points on (Ω_2) such that the chords AB, CD pass through M, N, respectively. Prove that if AC, BD, MN have a common point K, then TK is the angle bisector of $\angle MTN$.

* (Ω_2) is bigger than (Ω_1)

3 Let the integer $n \geq 2$, and $x_1, x_2, \cdots, x_n$ be positive real numbers such that $\sum_{i=1}^{n} x_i = 1$ .Prove that

$$\left(\sum_{i=1}^{n} \frac{1}{1-x_i}\right)\left(\sum_{1 \leq i < j \leq n} x_i x_j\right) \leq \frac{n}{2}.$$

4 For 100 straight lines on a plane, let T be the set of all right-angled triangles bounded by some 3 lines. Determine, with proof, the maximum value of $|T|$.

Day 2

5 Let a, b, c, d are lengths of the sides of a convex quadrangle with the area equal to S, set $S = \{x_1, x_2, x_3, x_4\}$ consists of permutations x_i of (a, b, c, d). Prove that

$$S \leq \frac{1}{2}(x_1 x_2 + x_3 x_4).$$

6 For a sequence $a_1, a_2, ..., a_m$ of real numbers, define the following sets

$$A = \{a_i | 1 \leq i \leq m\} \text{ and } B = \{a_i + 2a_j | 1 \leq i, j \leq m, i \neq j\}$$

Let n be a given integer, and $n > 2$. For any strictly increasing arithmetic sequence of positive integers, determine, with proof, the minimum number of elements of set $A \triangle B$, where $A \triangle B = (A \cup B) \setminus (A \cap B)$.

7 Let $a \in (0,1)$, $f(z) = z^2 - z + a$, $z \in \mathbb{C}$. Prove the following statement holds:
For any complex number z with $|z| \geq 1$, there exists a complex number z_0 with $|z_0| = 1$, such that $|f(z_0)| \leq |f(z)|$.

---

8 Let k be a positive integer, and $n = \left(2^k\right)!$ .Prove that $\sigma(n)$ has at least a prime divisor larger than 2^k, where $\sigma(n)$ is the sum of all positive divisors of n.
