Mathematical Olympiad 2009

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– You are given three hours to solve all problems. Each item is worth eight points.

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1 The sequence $a_0, a_1, a_2, ...$ of real numbers satisfies the recursive relation

$$n(n+1)a_{n+1} + (n-2)a_{n-1} = n(n-1)a_n$$

for every positive integer n, where $a_0 = a_1 = 1$. Calculate the sum

$$\frac{a_0}{a_1} + \frac{a_1}{a_2} + ... + \frac{a_{2008}}{a_{2009}}$$

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2 **(a)** Find all pairs (n, x) of positive integers that satisfy the equation $2^n + 1 = x^2$.

(b) Find all pairs (n, x) of positive integers that satisfy the equation $2^n = x^2 + 1$.

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3 Each point of a circle is colored either red or blue.

(a) Prove that there always exists an isosceles triangle inscribed in this circle such that all its vertices are colored the same.

(b) Does there always exist an equilateral triangle inscribed in this circle such that all its vertices are colored the same?

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4 Let k be a positive real number such that

$$\frac{1}{k+a} + \frac{1}{k+b} + \frac{1}{k+c} \leq 1$$

for any positive positive real numbers a, b and c with $abc = 1$. Find the minimum value of k.

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5 Segments AC and BD intersect at point P such that $PA = PD$ and $PB = PC$. Let E be the foot of the perpendicular from P to the line CD. Prove that the line PE and the perpendicular bisectors of the segments PA and PB are concurrent.
