Korea National Olympiad 2015

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Day 1

1 For a positive integer m, prove that the number of pairs of positive integers (x, y) which satisfies the following two conditions is even or 0.

(i): $x^2 - 3y^2 + 2 = 16m$

(ii): $2y \le x - 1$

2 Let the circumcircle of $\triangle ABC$ be ω. A point D lies on segment BC, and E lies on segment AD. Let ray $AD \cap \omega = F$. A point M, which lies on ω, bisects AF and it is on the other side of C with respect to AF. Ray $ME \cap \omega = G$, ray $GD \cap \omega = H$, and $MH \cap AD = K$. Prove that B, E, C, K are cyclic.

3 Reals a, b, c, x, y satisfies $a^2 + b^2 + c^2 + x^2 + y^2 = 1$. Find the maximum value of

$$(ax + by)^2 + (bx + cy)^2$$

4 For positive integers n, k, l, we define the number of l-tuples of positive integers $(a_1, a_2, \cdots a_l)$ satisfying the following as $Q(n, k, l)$.

(i): $n = a_1 + a_2 + \cdots + a_l$

(ii): $a_1 > a_2 > \cdots > a_l > 0$.

(iii): a_l is an odd number.

(iv): There are k odd numbers out of a_i.

For example, from $9 = 8 + 1 = 6 + 3 = 6 + 2 + 1$, we have $Q(9, 1, 1) = 1$, $Q(9, 1, 2) = 2$, $Q(9, 1, 3) = 1$.

Prove that if $n > k^2$, $\sum_{l=1}^{n} Q(n, k, l)$ is 0 or an even number.

Day 2

1 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all reals x, y, z, we have

$$(f(x) + 1)(f(y) + f(z)) = f(xy + z) + f(xz - y)$$

2 An isosceles trapezoid $ABCD$, inscribed in ω, satisfies $AB = CD, AD < BC, AD < CD$.
A circle with center D and passing A hits BD, CD, ω at $E, F, P(\neq A)$, respectively.
Let $AP \cap EF = Q$, and ω meet CQ and the circumcircle of $\triangle BEQ$ at $R(\neq C), S(\neq B)$, respectively.
Prove that $\angle BER = \angle FSC$.

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3 A positive integer n is given. If there exists sets $F_1, F_2, \cdots F_m$ satisfying the following conditions, prove that $m \leq n$. (For sets A, B, $|A|$ is the number of elements of A. $A - B$ is the set of elements that are in A but not B. min(x, y) is the number that is not larger than the other.)

(i): For all $1 \leq i \leq m$, $F_i \subseteq \{1, 2, \cdots, n\}$

(ii): For all $1 \leq i < j \leq m$, min$(|F_i - F_j|, |F_j - F_i|) = 1$

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4 For a positive integer n, $a_1, a_2, \cdots a_k$ are all positive integers without repetition that are not greater than n and relatively prime to n. If $k > 8$, prove the following.

$$\sum_{i=1}^{k} |a_i - \frac{n}{2}| < \frac{n(k-4)}{2}$$
