National Math Olympiad (3rd round) 2015

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–	Algebra
1	x,y,z are three real numbers inequal to zero satisfying $x+y+z=xyz$. Prove that $$\sum(\frac{x^2-1}{x})^2\geq 4$$ *Proposed by Amin Fathpour*
2	Prove that there are no functions $f,g:\mathbb{R}\rightarrow\mathbb{R}$ such that $\forall x,y\in\mathbb{R}: f(x^2+g(y))-f(x^2)+g(y)-g(x)\leq 2y$ and $f(x)\geq x^2$. *Proposed by Mohammad Ahmadi*
3	Does there exist an irreducible two variable polynomial $f(x,y)\in\mathbb{Q}[x,y]$ such that it has only four roots $(0,1),(1,0),(0,-1),(-1,0)$ on the unit circle.
4	$p(x)\in\mathbb{C}[x]$ is a polynomial such that: $\forall z\in\mathbb{C}, \|z\|=1\Longrightarrow p(z)\in\mathbb{R}$ Prove that $p(x)$ is constant.
5	Find all polynomials $p(x)\in\mathbb{R}[x]$ such that for all $x\in\mathbb{R}$: $p(5x)^2-3=p(5x^2+1)$ such that: $a)p(0)\neq 0$ $b)p(0)=0$
6	$a_1,a_2,\dots,a_n>0$ are positive real numbers such that $\sum_{i=1}^n\frac{1}{a_i}=n$ prove that: $\sum_{i<j}\left(\frac{a_i-a_j}{a_i+a_j}\right)^2\leq\frac{n^2}{2}\left(1-\frac{n}{\sum_{i=1}^n a_i}\right)$
–	Number Theory
1	Prove that there are infinitely natural numbers n such that n can't be written as a sum of two positive integers with prime factors less than 1394.
2	$M_0\subset\mathbb{N}$ is a non-empty set with a finite number of elements. Ali produces sets $M_1,M_2,...,M_n$ in the following order. In step n, Ali chooses an element of M_{n-1} like b_n and defines M_n as $$M_n=\{b_nm+1\|m\in M_{n-1}\}$$

Prove that at some step Ali reaches a set which no element of it divides another element of it.

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3 Let $p > 5$ be a prime number and $A = \{b_1, b_2, \dots, b_{\frac{p-1}{2}}\}$ be the set of all quadratic residues modulo p, excluding zero. Prove that there doesn't exist any natural a, c satisfying $(ac, p) = 1$ such that set $B = \{ab_1 + c, ab_2 + c, \dots, ab_{\frac{p-1}{2}} + c\}$ and set A are disjoint modulo p.

This problem was proposed by Amir Hossein Pooya.

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4 a, b, c, d, k, l are positive integers such that for every natural number n the set of prime factors of $n^k + a^n + c, n^l + b^n + d$ are same. prove that $k = l, a = b, c = d$.

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5 $p > 30$ is a prime number. Prove that one of the following numbers is in form of $x^2 + y^2$.

$$p + 1, 2p + 1, 3p + 1, ..., (p - 3)p + 1$$

– Geometry

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1 Let $ABCD$ be the trapezoid such that $AB \parallel CD$. Let E be an arbitrary point on AC. point F lies on BD such that $BE \parallel CF$. Prove that circumcircles of $\triangle ABF, \triangle BED$ and the line AC are concurrent.

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2 Let ABC be a triangle with orthocenter H and circumcenter O. Let K be the midpoint of AH. point P lies on AC such that $\angle BKP = 90°$. Prove that $OP \parallel BC$.

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3 Let ABC be a triangle. consider an arbitrary point P on the plain of $\triangle ABC$. Let R, Q be the reflections of P wrt AB, AC respectively. Let $RQ \cap BC = T$. Prove that $\angle APB = \angle APC$ if and if only $\angle APT = 90°$.

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4 Let ABC be a triangle with incenter I. Let K be the midpoint of AI and $BI \cap \odot(\triangle ABC) = M, CI \cap \odot(\triangle ABC) = N$. points P, Q lie on AM, AN respectively such that $\angle ABK = \angle PBC, \angle ACK = \angle QCB$. Prove that P, Q, I are collinear.

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5 Let ABC be a triangle with orthocenter H and circumcenter O. Let R be the radius of circumcircle of $\triangle ABC$. Let A', B', C' be the points on $\overrightarrow{AH}, \overrightarrow{BH}, \overrightarrow{CH}$ respectively such that $AH.AA' = R^2, BH.BB' = R^2, CH.CC' = R^2$. Prove that O is incenter of $\triangle A'B'C'$.
