Estonia Team Selection Test 1996

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by IstekOlympiadTeam

– Day 1

1 Suppose that x, y and $\frac{x^2+y^2+6}{xy}$ are positive integers . Prove that $\frac{x^2+y^2+6}{xy}$ is a perfect cube.

2 Let a, b, c be the sides of a triangle, α, β, γ the corresponding angles and r the inradius. Prove that

$$a \cdot sin\alpha + b \cdot sin\beta + c \cdot sin\gamma \geq 9r$$

3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ which satisfy for all x:

(i) $f(x) = -f(-x)$;

(ii) $f(x + 1) = f(x) + 1$;

(iii) $f\left(\frac{1}{x}\right) = \frac{1}{x^2} f(x)$ for $x \neq 0$

– Day 2

1 Prove that the polynomial $P_n(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ has no real zeros if n is even and has exatly one real zero if n is odd

2 Let H be the orthocenter of an obtuse triangle ABC and $A_1B_1C_1$ arbitrary points on the sides BC, AC, AB respectively.Prove that the tangents drawn from H to the circles with diametrs AA_1, BB_1, CC_1 are equal.

3 Each face of a cube is divided into n^2 equal squares. The vertices of the squares are called *nodes*, so each face has $(n + 1)^2$ nodes.

(a) If $n = 2$,does there exist a closed polygonal line whose links are sids of the squares and which passes through each node exactly once?

(b) Prove that, for each n, such a polygonal line divides the surface area of the cube into two equal parts