Estonia Team Selection Test 1997

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– Day 1

1 In a triangle ABC points A_1, B_1, C_1 are the midpoints of BC, CA, AB respectively,and A_2, B_2, C_2 are the midpoints of the altitudes from A, B, C respectively. Show that the lines A_1A_2, B_1B_2, C_1, C_2 are concurrent.

2 Prove that for all positive real numbers $a_1, a_2, \cdots a_n$

$$\frac{1}{\frac{1}{1+a_1}+\frac{1}{1+a_2}+\cdots+\frac{1}{1+a_n}}-\frac{1}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}\geq\frac{1}{n}$$

When does the inequality hold?

3 There are n boyfriend-girlfriend pairs at a party. Initially all the girls sit at a round table. For the first dance, each boy invites one of the girls to dance with.After each dance, a boy takes the girl he danced with to her seat, and for the next dance he invites the girl next to her in the counterclockwise direction. For which values of n can the girls be selected in such a way that in every dance at least one boy danced with his girlfriend, assuming that there are no less than n dances?

– Day 2

1 (a) Is it possible to partition the segment $[0, 1]$ into two sets A and B and to define a continuous function f such that for every $x \in A$ $f(x)$ is in B, and for every $x \in B$ $f(x)$ is in A?

(b) The same question with $[0, 1]$ replaced by $[0, 1)$.

2 A quadrilateral $ABCD$ is inscribed in a circle. On each of the sides AB, BC, CD, DA one erects a rectangle towards the interior of the quadrilateral, the other side of the rectangle being equal to $CD, DA, AB, BC,$ respectively. Prove that the centers of these four rectangles are vertices of a rectangle.

3 It is known that for every integer $n > 1$ there is a prime number among the numbers $n+1, n+2, ..., 2n-1$. Determine all positive integers n with the following property: Every integer $m > 1$ less than n and coprime to n is prime.