**Mexico National Olympiad 2015**

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| – | Day 1 |
| **1** | Let $ABC$ be an acuted-angle triangle and let $H$ be it's orthocenter. Let $PQ$ be a segment through $H$ such that $P$ lies on $AB$ and $Q$ lies on $AC$ and such that $\angle PHB = \angle CHQ$. Finally, in the circumcircle of $\triangle ABC$, consider $M$ such that $M$ is the mid point of the arc $BC$ that doesn't contain $A$. Prove that $MP = MQ$<br><br>Proposed by Eduardo Velasco/Marco Figueroa |
| **2** | Let $n$ be a positive integer and let $k$ be an integer between $1$ and $n$ inclusive. There is a white board of $n \times n$. We do the following process. We draw $k$ rectangles with integer sides lenghts and sides parallel to the ones of the $n \times n$ board, and such that each rectangle covers the top-right corner of the $n \times n$ board. Then, the $k$ rectangles are painted of black. This process leaves a white figure in the board.<br><br>How many different white figures are possible to do with $k$ rectangles that can't be done with less than $k$ rectangles?<br><br>Proposed by David Torres Flores |
| **3** | Let $\mathbb{N} = \{1, 2, 3, ...\}$ be the set of positive integers. Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be a function that gives a positive integer value, to every positive integer. Suppose that $f$ satisfies the following conditions:<br><br>$f(1) = 1$ $f(a + b + ab) = a + b + f(ab)$<br><br>Find the value of $f(2015)$<br><br>Proposed by Jose Antonio Gomez Ortega |
| – | Day 2 |
| **4** | Let $n$ be a positive integer. Mary writes the $n^3$ triples of not necessarily distinct integers, each between $1$ and $n$ inclusive on a board. Afterwards, she finds the greatest (possibly more than one), and erases the rest. For example, in the triple $(1, 3, 4)$ she erases the numbers 1 and 3, and in the triple $(1, 2, 2)$ she erases only the number 1,<br><br>Show after finishing this process, the amount of remaining numbers on the board cannot be a perfect square. |

 

**5** Let $I$ be the incenter of an acute-angled triangle $ABC$. Line $AI$ cuts the circumcircle of $BIC$ again at $E$. Let $D$ be the foot of the altitude from $A$ to $BC$, and let $J$ be the reflection of $I$ across $BC$. Show $D$, $J$ and $E$ are collinear.

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**6** Let $n$ be a positive integer and let $d_1, d_2, \ldots, d_k$ be its positive divisors. Consider the number

$$f(n) = (-1)^{d_1} d_1 + (-1)^{d_2} d_2 + \cdots + (-1)^{d_k} d_k$$

Assume $f(n)$ is a power of 2. Show if $m$ is an integer greater than 1, then $m^2$ does not divide $n$.

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