Japan MO Finals 2021

www.artofproblemsolving.com/community/c1944236

by maple116

1 Find all functions from positive integers to themselves, such that for any positive integers m, n the two conditions below are equivalent:
n divides m.
$f(n)$ divides $f(m) - n$.

---

2 Let $n \geq 2$ be an integer. Players A and B play a game using $n \times 2021$ grid of square unit cells. Firstly, A paints each cell either black of white. B places a piece in one of the cells in the uppermost row, and designates one of the cells in the lowermost row as the *goal*. Then, A repeats the following operation $n - 1$ times:
When the cell with the piece is painted white, A moves the piece to the cell one below.
Otherwise, A moves the piece to the next cell on the left or right, and then to the cell one below.
Find the minimum possible value of n such that A can always move a piece to the *goal*, regardless of B's choice.

---

3 Points D, E on the side AB, AC of an acute-angled triangle ABC respectively satisfy $BD = CE$. Furthermore, points P on the segmet DE and Q on the arc BC of the circle ABC not containing A satisfy $BP : PC = EQ : QD$. Points A, B, C, D, E, P, Q are pairwise distinct.
Prove that $\angle BPC = \angle BAC + \angle EQD$ holds.

---

4 Let $a_1, a_2, \ldots, a_{2021}$ be 2021 integers which satisfy

$$a_{n+5} + a_n > a_{n+2} + a_{n+3}$$

for all integers $n = 1, 2, \ldots, 2016$. Find the minimum possible value of the difference between the maximum value and the minimum value among $a_1, a_2, \ldots, a_{2021}$.

---

5 Let n be a positive integer. Find all integers k among $1, 2, \ldots, 2n^2$ which satisfy the following condition:
There is a $2n \times 2n$ grid of square unit cells. When k different cells are painted black while the other cells are painted white, the minimum possible number of 2×2 squares that contain both black and white cells is $2n - 1$.
