IMC 1995

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–	Day 1
1	Let X be a invertible matrix with columns $X_1, X_2..., X_n$. Let Y be a matrix with columns $X_2, X_3, ..., X_n, 0$. Show that the matrices $A = YX^{-1}$ and $B = X^{-1}Y$ have rank $n-1$ and have only 0s for eigenvalues.
2	Let f be a continuous function on $[0, 1]$ such that for every $x \in [0, 1]$, we have $\int_x^1 f(t)dt \geq \frac{1-x^2}{2}$. Show that $\int_0^1 f(t)^2dt \geq \frac{1}{3}$.
3	Let f be twice continuously differentiable on $(0, \infty)$ such that $\lim_{x\to 0^+} f'(x) = -\infty$ and $\lim_{x\to 0^+} f''(x) = \infty$. Show that $$\lim_{x\to 0^+} \frac{f(x)}{f'(x)} = 0.$$
4	Let $F : (1, \infty) \to \mathbb{R}$ be the function defined by $$F(x) = \int_x^{x^2} \frac{dt}{\ln(t)}.$$ Show that F is injective and find the set of values of F.
5	Let A and B be real $n \times n$ matrices. Assume there exist $n+1$ different real numbers $t_1, t_2, \dots, t_{n+1}$ such that the matrices $$C_i = A + t_iB,\ i = 1, 2, \dots, n+1$$ are nilpotent. Show that both A and B are nilpotent.
6	Let $p > 1$. Show that there exists a constant $K_p > 0$ such that for every $x, y \in \mathbb{R}$ with $\lvert x\rvert^p + \lvert y\rvert^p = 2$, we have $$(x-y)^2 \leq K_p(4-(x+y)^2).$$
–	Day 2
7	Let A be a 3×3 real matrix such that the vectors Au and u are orthogonal for every column vector $u \in \mathbb{R}^3$. Prove that: a) $A^T = -A$.

b) there exists a vector $v \in \mathbb{R}^3$ such that $Au = v \times u$ for every $u \in \mathbb{R}^3$,
where $v \times u$ denotes the vector product in $\mathbb{R}^3$.

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8 Let $(b_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers such that $b_0 = 1$, $b_n = 2 + \sqrt{b_{n-1}} - 2\sqrt{1+\sqrt{b_{n-1}}}$. Calculate

$$\sum_{n=1}^{\infty} b_n 2^n.$$

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9 Let all roots of an n-th degree polynomial $P(z)$ with complex coefficients lie on the unit circle in the complex plane. Prove that all roots of the polynomial

$$2zP'(z) - nP(z)$$

lie on the same circle.

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10 a) Prove that for every $\epsilon > 0$ there is a positive integer n and real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$\max_{x\in[-1,1]} |x - \sum_{k=1}^{n} \lambda_k x^{2k+1}| < \epsilon.$$

b) Prove that for every odd continuous function f on $[-1,1]$ and for every $\epsilon > 0$ there is a positive integer n and real numbers $\mu_1, \ldots, \mu_n$ such that

$$\max_{x\in[-1,1]} |f(x) - \sum_{k=1}^{n} \mu_k x^{2k+1}| < \epsilon.$$

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11 a) Prove that every function of the form

$$f(x) = \frac{a_0}{2} + \cos(x) + \sum_{n=2}^{N} a_n \cos(nx)$$

with $|a_0| < 1$ has positive as well as negative values in the period $[0, 2\pi)$.
b) Prove that the function

$$F(x) = \sum_{n=1}^{100} \cos(n^{\frac{3}{2}} x)$$

has at least 40 zeroes in the interval $(0, 1000)$.

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12 Suppose that $(f_n)_{n=1}^{\infty}$ is a sequence of continuous functions on the interval $[0,1]$ such that

$$\int_0^1 f_m(x)f_n(x)dx = \begin{cases} 1 & \text{if } n=m \\ 0 & \text{if } n \neq m \end{cases}$$

and $\sup\{|f_n(x)| : x \in [0,1] \text{ and } n = 1,2,\dots\} < \infty$.
Show that there exists no subsequence (f_{n_k}) of (f_n) such that $\lim_{k\to\infty} f_{n_k}(x)$ exist for all $x \in [0,1]$.