Final Round - Switzerland 2021

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–	Day 1
1	Let (m, n) be pair of positive integers. Julia has carefully planted m rows of n dandelions in an $m \times n$ array in her back garden. Now, Jana un Viviane decides to play a game with a lawnmower they just found. Taking alternating turns and starting with Jana, they can now mow down all the dandelions in a straight horizontal or vertical line (and they must mow down at least one dandelion). The winner is the player who mows down the final dandelion. Determine all pairs of (m, n) for which Jana has a winning strategy.
2	Let $\triangle ABC$ be an acute triangle with $AB = AC$ and let D be a point on the side BC. The circle with centre D passing through C intersects $\odot(ABD)$ at points P and Q, where Q is the point closer to B. The line BQ intersects AD and AC at points X and Y respectively. Prove that quadrilateral $PDXY$ is cyclic.
3	Find all finite sets S of positive integers with at least 2 elements, such that if $m > n$ are two elements of S, then $$\frac{n^2}{m-n}$$ is also an element of S.
4	Suppose that a, b, c, d are positive real numbers satisfying $(a + c)(b + d) = ac + bd$. Find the smallest possible value of $$\frac{a}{b} + \frac{b}{c} + \frac{c}{d} + \frac{d}{a}.$$ *Israel*
–	Day 2
5	For which integers $n \geq 2$ can we arrange numbers $1, 2, \ldots, n$ in a row, such that for all integers $1 \leq k \leq n$ the sum of the first k numbers in the row is divisible by k?
6	Let $\mathbb{N}$ be the set of positive integers. Let $f : \mathbb{N} \to \mathbb{N}$ be a function such that for every positive integer $n \in \mathbb{N}$ $$f(n) - n < 2021 \quad \text{and} \quad f^{f(n)}(n) = n$$ Prove that $f(n) = n$ for infinitely many $n \in \mathbb{N}$

7 Let $m \geq n$ be positive integers. Frieder is given mn posters of Linus with different integer dimensions of $k \times l$ with $1 \geq k \geq m$ and $1 \geq l \geq n$. He must put them all up one by one on his bedroom wall without rotating them. Every time he puts up a poster, he can either put it on an empty spot on the wall or on a spot where it entirely covers a single visible poster and does not overlap any other visible poster. Determine the minimal area of the wall that will be covered by posters.

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8 Let $\triangle ABC$ be a triangle with $AB = AC$ and $\angle BAC = 20^\circ$. Let D be point on the side AB such that $\angle BCD = 70^\circ$. Let E be point on the side AC such that $\angle CBE = 60^\circ$. Determine the value of angle $\angle CDE$.
