Kosovo National Mathematical Olympiad 2021

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by bsf714, Com10atorics, geometry6

–	Grade 9
1	Find all natural two digit numbers such that when you substract by seven times the sum of its digit from the number you get a prime number.
2	Dua has all the odd natural numbers less than 20. Asija has all the even numbers less than 21. They play the following game. In each round, they take a number from each other and after every round, they may fix two or more consecutive numbers so that their opponent cannot take these fixed numbers in the next round. The game is won by the player who attains 10 consecutive numbers first. Does either player have a winning strategy?
3	Find all real numbers a, b, c and d such that: $a^2+b^2+c^2+d^2=a+b+c+d-ab=3.$
4	Let $ABCDE$ be a convex pentagon such that: $\angle ABC=90, \angle BCD=135, \angle DEA=60$ and $AB=BC=CD=DE$. Find angle $\angle DAE$.
–	Grade 10
1	Nine weights are placed in a scale with the respective values $1kg, 2kg, ..., 9kg$. In how many ways can we place six weights in the left side and three weights in the right side such that the right side is heavier than the left one?
2	Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real numbers x, y: $f(x)f(y)+f(xy)\leq x+y$.
3	Prove that for any natural numbers a, b, c and d there exist infinetly natural numbers n such that $a^n+b^n+c^n+d^n$ is composite.
4	Let M be the midpoint of segment BC of $\triangle ABC$. Let D be a point such that $AD=AB$, $AD\perp AB$ and points C and D are on different sides of AB. Prove that: $$\sqrt{AB\cdot AC+BC\cdot AM}\geq\frac{\sqrt{2}}{2}CD.$$
–	Grade 11

1 There are 9 point in the Cartezian plane with coordinates $(0,0),(0,1),(0,2),(1,0),(1,1),(1,2),(2,0),(2,1),($
Some points are coloured in red and the others in blue. Prove that for any colouring of the points we can always find a right isosceles triangle whose vertexes have the same colour.

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2 Does there exist a natural number n such that $n!$ ends with exactly 2021 zeros?

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3 Let a, b and c be positive real numbers such that $a^5+b^5+c^5=ab^2+bc^2+ca^2$.
Prove the inequality:

$$\frac{a^2+b^2}{b}+\frac{b^2+c^2}{c}+\frac{c^2+a^2}{a}\geq 2(ab+bc+ca).$$

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4 Let ABC be a triangle with $AB<AC$. Let D be the point where the bisector of angle $\angle BAC$ touches BC and let D' be the reflection of D in the midpoint of BC. Let X be the intersection of the bisector of angle $\angle BAC$ with the line parallel to AB that passes through D'. Prove that the line AC is tangent with the circumscribed circle of triangle XCD'

– Grade 12

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1 Each of the spots in a 8×8 chessboard is occupied by either a black or white horse. At most how many black horses can be on the chessboard so that none of the horses attack more than one black horse?

Remark: A black horse could attack another black horse.

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2 Find all functions $f:\mathbb{R}\to\mathbb{R}$ so that the following relation holds for all $x,y\in\mathbb{R}$.

$$f(f(x)f(y)-1)=xy-1$$

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3 Let ABC be a triangle and let O be the centre of its circumscribed circle. Points X, Y which are neither of the points A, B or C, lie on the circumscribed circle and are so that the angles XOY and BAC are equal (with the same orientation). Show that the orthocentre of the triangle that is formed by the lines BY, CX and XY is a fixed point.

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4 Let $P(x)$ be a polynomial with integer coefficients. We will denote the set of all prime numbers by $\mathbb{P}$. Show that the set $\mathbb{S} := \{p\in\mathbb{P} : \exists\, n \text{ s.t. } p\mid P(n)\}$ is finite if and only if $P(x)$ is a non-zero constant polynomial.
