IMC 1996

www.artofproblemsolving.com/community/c1960552

by Tintarn, sqrtX

–	Day 1
1	Let $A=(a_{ij})\in M_{(n+1)\times(n+1)}(\mathbb{R})$ with $a_{ij}=a+\|i-j\|d$, where a and d are fixed real numbers. Calculate $\det(A)$.
2	Evaluate the definite integral $$\int_{-\pi}^{\pi}\frac{\sin nx}{(1+2^x)\sin x}dx,$$ where n is a natural number.
3	The linear operator A on a finite-dimensional vector space V is called an involution if $A^2=I$, where I is the identity operator. Let $\dim V=n$. i) Prove that for every involution A on V, there exists a basis of V consisting of eigenvectors of A. ii) Find the maximal number of distinct pairwise commuting involutions on V.
4	Let $a_1=1$, $a_n=\frac{1}{n}\sum_{k=1}^{n-1}a_ka_{n-k}$ for $n\geq 2$. Show that i) $\limsup_{n\to\infty}\|a_n\|^{\frac{1}{n}}<2^{-\frac{1}{2}}$; ii) $\limsup_{n\to\infty}\|a_n\|^{\frac{1}{n}}\geq\frac{2}{3}$
5	i) Let a,b be real numbers such that $b\leq 0$ and $1+ax+bx^2\geq 0$ for every $x\in[0,1]$. Prove that $$\lim_{n\to\infty}n\int_0^1(1+ax+bx^2)^n dx=\begin{cases}-\frac{1}{a} & \text{if } a<0,\\ \infty & \text{if } a\geq 0.\end{cases}$$ ii) Let $f:[0,1]\to[0,\infty)$ be a function with a continuous second derivative and let $f''(x)\leq 0$ for every $x\in[0,1]$. Suppose that $L=\lim_{n\to\infty}n\int_0^1(f(x))^n dx$ exists and $0<L<\infty$. Prove that f' has a constant sign and $\min_{x\in[0,1]}\|f'(x)\|=L^{-1}$.
6	Upper content of a subset E of the plane $\mathbb{R}^2$ is defined as $$\mathcal{C}(E)=\inf\{\sum_{i=1}^{n}\text{diam}(E_i)\}$$ where $\inf$ is taken over all finite families of sets $E_1,\dots,E_n$ $n\in\mathbb{N}$, in $\mathbb{R}^2$ such that $E\subset\bigcup_{i=1}^{n}E_i$.

Lower content of E is defined as

$$\mathcal{K}(E) = \sup\{\text{length}(L) |\ L \text{ is a closed line segment onto which } E \text{ can be contracted}\}$$

.

Prove that
i) $\mathcal{C}(L) = \text{length}(L)$ if L is a closed line segment;
ii) $\mathcal{C}(E) \geq \mathcal{K}(E)$;
iii) the equality in ii) is not always true even if E is compact.

– Day 2

7 Prove that if $f : [0,1] \to [0,1]$ is a continuous function, then the sequence of iterates $x_{n+1} = f(x_n)$ converges if and only if

$$\lim_{n\to\infty} (x_{n+1} - x_n) = 0$$

8 Let θ be a positive real number. Show that if $k \in \mathbb{N}$ and both $\cosh k\theta$ and $\cosh(k+1)\theta$ are rational, then so is $\cosh\theta$.

9 Let G be the subgroup of $GL_2(\mathbb{R})$ generated by A and B, where

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

.

Let H consist of the matrices $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ in G for which $a_{11} = a_{22} = 1$.
a) Show that H is an abelian subgroup of G.
b) Show that H is not finitely generated.

10 Let B be a bounded closed convex symmetric (with respect to the origin) set in $\mathbb{R}^2$ with boundary Γ. Let B have the property that the ellipse of maximal area contained in B is the disc D of radius 1 centered at the origin with boundary C. Prove that $A \cap \Gamma \neq \emptyset$ for any arc A of C of length $l(A) \geq \frac{\pi}{2}$.

11 i) Prove that

$$\lim_{x\to\infty} \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} = \frac{1}{2}$$

.

ii) Prove that there is a positive constant c such that for every $x \in [1, \infty)$ we have

$$\left| \sum_{n=1}^{\infty} \frac{nx}{(n^2+x)^2} - \frac{1}{2} \right| \leq \frac{c}{x}$$

12 i) Prove that for every sequence $(a_n)_{n\in\mathbb{N}}$, such that $a_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a_n < \infty$, we have

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{\frac{1}{n}} < e\sum_{n=1}^{\infty} a_n.$$

ii) Prove that for every $\epsilon > 0$ there exists a sequence $(b_n)_{n\in\mathbb{N}}$ such that $b_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} b_n < \infty$ and

$$\sum_{n=1}^{\infty}(b_1b_2\cdots b_n)^{\frac{1}{n}} > (e-\epsilon)\sum_{n=1}^{\infty} b_n.$$