National Olympiad Second Round 2020

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Day 1 March 6th, 2021

1 Let $n > 1$ be an integer and $X = \{1, 2, \cdots, n^2\}$. If there exist x, y such that $x^2 \mid y$ in all subsets of X with k elements, find the least possible value of k.

2 Let P be an interior point of acute triangle ΔABC, which is different from the orthocenter. Let D and E be the feet of altitudes from A to BP and CP, and let F and G be the feet of the altitudes from P to sides AB and AC. Denote by X the midpoint of $[AP]$, and let the second intersection of the circumcircles of triangles ΔDFX and ΔEGX lie on BC. Prove that AP is perpendicular to BC or $\angle PBA = \angle PCA$.

3 If x, y, z are positive real numbers find the minimum value of

$$2\sqrt{(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)} - \sqrt{\left(1+\frac{x}{y}\right)\left(1+\frac{y}{z}\right)}$$

Day 2 March 7th, 2021

4 Let p be a prime number such that $\frac{28^p-1}{2p^2+2p+1}$ is an integer. Find all possible values of number of divisors of $2p^2 + 2p + 1$.

5 Find all polynomials with real coefficients such that one can find an integer valued series $a_0, a_1, \ldots$ satisfying $\lfloor P(x) \rfloor = a_{\lfloor x^2 \rfloor}$ for all x real numbers.

6 2021 points are given on a circle. Each point is colored by one of the $1, 2, \cdots, k$ colors. For all points and colors $1 \leq r \leq k$, there exist an arc such that at least half of the points on it are colored with r. Find the maximum possible value of k.