Tournament Of Towns 1990

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by parmenides51

– Spring 1990

– Junior

– O Level

(243) 1 For every natural number n prove that

$$\left(1+\frac{1}{2}+...+\frac{1}{n}\right)^2+\left(\frac{1}{2}+...+\frac{1}{n}\right)^2+...+\left(\frac{1}{n-1}+\frac{1}{2}\right)^2+\left(\frac{1}{n}\right)^2=2n-\left(1+\frac{1}{2}+...+\frac{1}{n}\right)$$

(S. Manukian, Yerevan)

(244) 2 Two circles c and d are situated in the plane each outside the other. The points C and D are located on circles c and d respectively, so as to be as far apart as possible. Two smaller circles are constructed inside c and d. Of these the first circle touches c and the two tangents drawn from C to d, while the second circle touches d and the two tangents from D to c. Prove that the small circles are equal.

(J. Tabov, Sofia)

(245) 3 Is it possible to put together 27 equal cubes, 9 red, 9 blue and 9 white, so as to obtain a big cube in which each row (parallel to an arbitrary edge of the cube) contains three cubes with exactly two different colours?

(S. Fomin, Leningrad)

(246) 4 A set of 61 coins that look alike is given. Two coins (whose weights are equal) are counterfeit. The other 59 (genuine) coins also have the same weight, but a different weight from that of the counterfeit coins. However it is not known whether it is the genuine coins or the counterfeit coins which are heavier. How can this question be resolved by three weighings on the one balance? (It is not required to separate the counterfeit coins from the genuine ones.)

(D. Fomin, Leningrad)

– A Level

(247) 1 Find the maximum number of parts into which the Oxy-plane can be divided by 100 graphs of different quadratic functions of the form $y = ax^2 + bx + c$.

(N.B. Vasiliev, Moscow)

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(248) 2 If a square is intersected by another square equal to it but rotated by 45^o around its centre, each side is divided into three parts in a certain ratio $a : b : a$ (which one can compute). Make the following construction for an arbitrary convex quadrilateral: divide each of its sides into three parts in this same ratio $a : b : a$, and draw a line through the two division points neighbouring each vertex. Prove that the new quadrilateral bounded by the four drawn lines has the same area as the original one.

(A. Savin, Moscow)

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(249) 3 Fifteen elephants stand in a row. Their weights are expressed by integer numbers of kilograms. The sum of the weight of each elephant (except the one on the extreme right) and the doubled weight of its right neighbour is exactly 15 tonnes. Determine the weight of each elephant.

(F.L. Nazarov)

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(250) 4 Let $ABCD$ be a rhombus and P be a point on its side BC. The circle passing through A, B, and P intersects BD once more at the point Q and the circle passing through C, P and Q intersects BD once more at the point R. Prove that A, R and P lie on the one straight line.

(D. Fomin, Leningrad)

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(251) 5 Find the number of pairs (m, n) of positive integers, both of which are ≤ 1000, such that $\frac{m}{n+1} < \sqrt{2} < \frac{m+1}{n}$

(recalling that $\sqrt{2} = 1.414213...$).

(D. Fomin, Leningrad)

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(252) 6 We call a collection of weights (each weighing an integer value) basic if their total weight equals 200 and each object of integer weight not greater than 200 can be balanced exactly with a uniquely determined set of weights from the collection. (Uniquely means that we are not concerned with order or which weights of equalc value are chosen to balance against a particular object, if in fact there is a choice.)
(a) Find an example of a basic collection other than the collection of 200 weights each of value 1.
(b) How many different basic collections are there?

(D. Fomin, Leningrad)

– Senior

– O Level

– A Level