Canada National Olympiad 2021

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by MortemEtInteritum

1 Let $ABCD$ be a trapezoid with AB parallel to CD, $|AB| > |CD|$, and equal edges $|AD| = |BC|$. Let I be the center of the circle tangent to lines AB, AC and BD, where A and I are on opposite sides of BD. Let J be the center of the circle tangent to lines CD, AC and BD, where D and J are on opposite sides of AC. Prove that $|IC| = |JB|$.

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2 Let $n \geq 2$ be some fixed positive integer and suppose that $a_1, a_2, \dots, a_n$ are positive real numbers satisfying $a_1 + a_2 + \cdots + a_n = 2^n - 1$.

Find the minimum possible value of

$$\frac{a_1}{1} + \frac{a_2}{1+a_1} + \frac{a_3}{1+a_1+a_2} + \cdots + \frac{a_n}{1+a_1+a_2+\cdots+a_{n-1}}$$

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3 At a dinner party there are N hosts and N guests, seated around a circular table, where $N \geq 4$. A pair of two guests will chat with one another if either there is at most one person seated between them or if there are exactly two people between them, at least one of whom is a host. Prove that no matter how the $2N$ people are seated at the dinner party, at least N pairs of guests will chat with one another.

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4 A function f from the positive integers to the positive integers is called *Canadian* if it satisfies

$$\gcd\left(f(f(x)), f(x+y)\right) = \gcd(x, y)$$

for all pairs of positive integers x and y.

Find all positive integers m such that $f(m) = m$ for all Canadian functions f.

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5 Nina and Tadashi play the following game. Initially, a triple (a, b, c) of nonnegative integers with $a + b + c = 2021$ is written on a blackboard. Nina and Tadashi then take moves in turn, with Nina first. A player making a move chooses a positive integer k and one of the three entries on the board; then the player increases the chosen entry by k and decreases the other two entries by k. A player loses if, on their turn, some entry on the board becomes negative.

Find the number of initial triples (a, b, c) for which Tadashi has a winning strategy.
