VI Caucasus Mathematical Olympiad

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by bigant146

–	Juniors
–	Day 1
1	Let a, b, c be real numbers such that $a^2+b=c^2$, $b^2+c=a^2$, $c^2+a=b^2$. Find all possible values of abc.
2	In a triangle ABC let K be a point on the median BM such that $CK = CM$. It appears that $\angle CBM = 2\angle ABM$. Prove that $BC = MK$.
3	We have $n > 2$ non-zero integers such that each one of them is divisible by the sum of the other $n-1$ numbers. Prove that the sum of all the given numbers is zero.
4	A square grid $2n \times 2n$ is constructed of matches (each match is a segment of length 1). By one move Peter can choose a vertex which (at this moment) is the endpoint of 3 or 4 matches and delete two matches whose union is a segment of length 2. Find the least possible number of matches that could remain after a number of Peter's moves.
–	Day 2
5	Let a, b, c be positive integers such that the product $$\gcd(a,b)\cdot\gcd(b,c)\cdot\gcd(c,a)$$ is a perfect square. Prove that the product $$\operatorname{lcm}(a,b)\cdot\operatorname{lcm}(b,c)\cdot\operatorname{lcm}(c,a)$$ is also a perfect square.
6	A row of 2021 balls is given. Pasha and Vova play a game, taking turns to perform moves; Pasha begins. On each turn a boy should paint a non-painted ball in one of the three available colors: red, yellow, or green (initially all balls are non-painted). When all the balls are colored, Pasha wins, if there are three consecutive balls of different colors; otherwise Vova wins. Who has a winning strategy?
7	An acute triangle ABC is given. Let AD be its altitude, let H and O be its orthocenter and its circumcenter, respectively. Let K be the point on the segment AH with $AK = HD$; let L be the point on the segment CD with $CL = DB$. Prove that line KL passes through O.

8 Let us call a set of positive integers *nice*, if its number of elements is equal to the average of all its elements. Call a number n *amazing*, if one can partition the set $\{1, 2, \ldots, n\}$ into nice subsets.

a) Prove that any perfect square is amazing.

b) Prove that there exist infinitely many positive integers which are not amazing.

– Seniors

– Day 1

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1 Integers from 1 to 100 are placed in a row in some order. Let us call a number *large-right*, if it is greater than each number to the right of it; let us call a number *large-left*, is it is greater than each number to the left of it. It appears that in the row there are exactly k large-right numbers and exactly k large-left numbers. Find the maximal possible value of k.

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2 See Juniors 3

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3 Let $n \geq 3$ be a positive integer. In the plane n points which are not all collinear are marked. Find the least possible number of triangles whose vertices are all marked.

(Recall that the vertices of a triangle are not collinear.)

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4 In an acute triangle ABC let AH_a and BH_b be altitudes. Let H_aH_b intersect the circumcircle of ABC at P and Q. Let A' be the reflection of A in BC, and let B' be the reflection of B in CA. Prove that A', B', P, Q are concyclic.

– Day 2

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5 A triangle Δ with sidelengths $a \leq b \leq c$ is given. It appears that it is impossible to construct a triangle from three segments whose lengths are equal to the altitudes of Δ. Prove that $b^2 > ac$.

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6 See Juniors 6

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7 4 tokens are placed in the plane. If the tokens are now at the vertices of a convex quadrilateral P, then the following move could be performed: choose one of the tokens and shift it in the direction perpendicular to the diagonal of P not containing this token; while shifting tokens it is prohibited to get three collinear tokens.

Suppose that initially tokens were at the vertices of a rectangle Π, and after a number of moves tokens were at the vertices of one another rectangle Π' such that Π' is similar to Π but not equal to Π. Prove that Π is a square.

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8 An infinite table whose rows and columns are numbered with positive integers, is given. For a sequence of functions $f_1(x), f_2(x), \ldots$ let us place the number $f_i(j)$ into the cell (i, j) of the table (for all $i, j \in \mathbb{N}$).
A sequence $f_1(x), f_2(x), \ldots$ is said to be *nice*, if all the numbers in the table are positive integers, and each positive integer appears exactly once. Determine if there exists a nice sequence of functions $f_1(x), f_2(x), \ldots$, such that each $f_i(x)$ is a polynomial of degree 101 with integer coefficients and its leading coefficient equals to 1.