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**–** level 2

**1** We say that a four-digit number $\overline{abcd}$, which starts at $a$ and ends at $d$, is *interchangeable* if there is an integer $n > 1$ such that $n \times \overline{abcd}$ is a four-digit number that begins with $d$ and ends with $a$. For example, $1009$ is interchangeable since $1009 \times 9 = 9081$. Find the largest interchangeable number.

**2** How many squares must be painted at least on a $5 \times 5$ board such that in each row, in each column and in each $2 \times 2$ square is there at least one square painted?

**3** We say that a positive integer is *quad-divi* if it is divisible by the sum of the squares of its digits, and also none of its digits is equal to zero.
a) Find a quad-divi number such that the sum of its digits is $24$.
b) Find a quad-divi number such that the sum of its digits is $1001$.

**4** In a triangle $ABC$, let $D$ and $E$ be points of the sides $BC$ and $AC$ respectively. Segments $AD$ and $BE$ intersect at $O$. Suppose that the line connecting midpoints of the triangle and parallel to $AB$, bisects the segment $DE$. Prove that the triangle $ABO$ and the quadrilateral $ODCE$ have equal areas.

**5** Rosa and Sara play with a triangle $ABC$, right at $B$. Rosa begins by marking two interior points of the hypotenuse $AC$, then Sara marks an interior point of the hypotenuse $AC$ different from those of Rosa. Then, from these three points the perpendiculars to the sides $AB$ and $BC$ are drawn, forming the following figure.
https://cdn.artofproblemsolving.com/attachments/9/9/c964bbacc4a5960bee170865cc43902410e50
png
Sara wins if the area of the shaded surface is equal to the area of the unshaded surface, in other case wins Rosa. Determine who of the two has a winning strategy.

**–** level 1

**1** Seven different positive integers are written on a sheet of paper. The result of the multiplication of the seven numbers is the cube of a whole number. If the largest of the numbers written on the sheet is $N$, determine the smallest possible value of $N$. Show an example for that value of $N$ and explain why $N$ cannot be smaller.

**2** In a sports competition in which several tests are carried out, only the three athletes $A, B, C$. In each event, the winner receives $x$ points, the second receives $y$ points, and the third receives

$z$ points. There are no ties, and the numbers $x, y, z$ are distinct positive integers with $x$ greater than $y$, and $y$ greater than $z$.
At the end of the competition it turns out that $A$ has accumulated $20$ points, $B$ has accumulated $10$ points and $C$ has accumulated $9$ points. We know that athlete $A$ was second in the 100-meter event. Determine which of the three athletes he was second in the jumping event.

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**3** Let $ABCD$ be a rhombus of sides $AB = BC = CD = DA = 13$. On the side $AB$ construct the rhombus $BAFC$ outside $ABCD$ and such that the side $AF$ is parallel to the diagonal $BD$ of $ABCD$. If the area of $BAFE$ is equal to $65$, calculate the area of $ABCD$.

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**4** Given a board of $3 \times 3$ you want to write the numbers $1, 2, 3, 4, 5, 6, 7, 8$ and a number in their boxes positive integer $M$, not necessarily different from the above. The goal is that the sum of the three numbers in each row be the same $a)$ Find all the values of $M$ for which this is possible. $b)$ For which of the values of $M$ found in $a)$ is it possible to arrange the numbers so that no only the three rows add the same but also the three columns add the same?

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**5** On the blackboard are written the $400$ integers $1, 2, 3, \cdots, 399, 400$. Luis erases $100$ of these numbers, then Martin erases another $100$. Martin wins if the sum of the $200$ erased numbers equals the sum of those not deleted; otherwise, he wins Luis. Which of the two has a winning strategy? What if Luis deletes $101$ numbers and Martín deletes $99$?
In each case, explain how the player with the winning strategy can ensure victory.

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