Brazil National Olympiad 2020, took place in 15-16 March 2021

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Day 1 Monday, March 15 (2021)

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1 Prove that there are positive integers $a_1, a_2, \dots, a_{2020}$ such that

$$\frac{1}{a_1}+\frac{1}{2a_2}+\frac{1}{3a_3}+\dots+\frac{1}{2020a_{2020}}=1.$$

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2 For a positive integer a, define $F_1^{(a)}=1$, $F_2^{(a)}=a$ and for $n>2$, $F_n^{(a)}=F_{n-1}^{(a)}+F_{n-2}^{(a)}$. A positive integer is *fibonatic* when it is equal to $F_n^{(a)}$ for a positive integer a and $n>3$. Prove that there are infintely many not *fibonatic* integers.

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3 Let r_A, r_B, r_C rays from point P. Define circles w_A, w_B, w_C with centers X, Y, Z such that w_a is tangent to r_B, r_C, w_B is tangent to r_A, r_C and w_C is tangent to r_A, r_B. Suppose P lies inside triangle XYZ, and let s_A, s_B, s_C be the internal tangents to circles w_B and w_C; w_A and w_C; w_A and w_B that do not contain rays r_A, r_B, r_C respectively. Prove that s_A, s_B, s_C concur at a point Q, and also that P and Q are isotomic conjugates.
PS: The rays can be lines and the problem is still true.

Day 2 Tuesday, March 16 (2021)

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4 Let ABC be a triangle. The ex-circles touch sides BC, CA and AB at points U, V and W, respectively. Be r_u a straight line that passes through U and is perpendicular to BC, r_v the straight line that passes through V and is perpendicular to AC and r_w the straight line that passes through W and is perpendicular to AB. Prove that the lines r_u, r_v and r_w pass through the same point.

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5 Let n and k be positive integers with $k \leq n$. In a group of n people, each one or always speak the truth or always lie. Arnaldo can ask questions for any of these people provided these questions are of the type: In set A, what is the parity of people who speak to true? , where A is a subset of size k of the set of n people. The answer can only be $even$ or odd.
a) For which values of n and k is it possible to determine which people speak the truth and which people always lie?
b) What is the minimum number of questions required to determine which people speak the truth and which people always lie, when that number is finite?

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6 Let $f(x) = 2x^2 + x - 1$, $f^0(x) = x$ and $f^{n+1}(x) = f(f^n(x))$ for all real x and $n \geq 0$ integer .
(a) Determine the number of real distinct solutions of the equation of $f^3(x) = x$.
(b) Determine, for each integer $n \geq 0$, the number of real distinct solutions of the equation $f^n(x) = 0$.

– level 2

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1 Let ABC be an acute triangle and AD a height. The angle bissector of $\angle DAC$ intersects DC at E. Let F be a point on AE such that BF is perpendicular to AE. If $\angle BAE = 45$, find $\angle BFC$.

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2 The following sentece is written on a board:

The equation $x^2 - 824x + \blacksquare 143 = 0$ has two integer solutions.

Where ■ represents algarisms of a blurred number on the board. What are the possible equations originally on the board?

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3 Consider an inifinte sequence $x_1, x_2, \dots$ of positive integers such that, for every integer $n \geq 1$:
-If x_n is even, $x_{n+1} = \dfrac{x_n}{2}$;
-If x_n is odd, $x_{n+1} = \dfrac{x_n - 1}{2} + 2^{k-1}$, where $2^{k-1} \leq x_n < 2^k$.
Determine the smaller possible value of x_1 for which 2020 is in the sequence.

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4 A positive integer is *brazilian* if the first digit and the last digit are equal. For instance, 4 and 4104 are brazilians, but 10 is not brazilian. A brazilian number is *superbrazilian* if it can be written as sum of two brazilian numbers. For instance, $101 = 99 + 2$ and $22 = 11 + 11$ are superbrazilians, but $561 = 484 + 77$ is not superbrazilian, because 561 is not brazilian. How many 4-digit numbers are superbrazilians?

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5 Let ABC be a triangle and M the midpoint of AB. Let circumcircles of triangles CMO and ABC intersect at K where O is the circumcenter of ABC. Let P be the intersection of lines OM and CK. Prove that $\angle PAK = \angle MCB$.

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6 Let k be a positive integer. Arnaldo and Bernaldo play a game in a table 2020×2020, initially all the cells are empty. In each round a player chooses a empty cell and put one red token or one blue token, Arnaldo wins if in some moment, there are k consecutive cells in the same row or column with tokens of same color, if all the cells have a token and there aren't k consecutive cells(row or column) with same color, then Bernaldo wins. If the players play alternately and Arnaldo goes first, determine for which values of k, Arnaldo has the winning strategy.
