China Team Selection Test 2021

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Test 1 Day 1

1 Given positive integers m and n. Let $a_{i,j}(1 \le i \le m, 1 \le j \le n)$ be non-negative real numbers, such that

$$a_{i,1} \ge a_{i,2} \ge \cdots \ge a_{i,n} \text{ and } a_{1,j} \ge a_{2,j} \ge \cdots \ge a_{m,j}$$

holds for all $1 \le i \le m$ and $1 \le j \le n$. Denote

$$X_{i,j} = a_{1,j} + \cdots + a_{i-1,j} + a_{i,j} + a_{i,j-1} + \cdots + a_{i,1},$$

$$Y_{i,j} = a_{m,j} + \cdots + a_{i+1,j} + a_{i,j} + a_{i,j+1} + \cdots + a_{i,n}.$$

Prove that

$$\prod_{i=1}^{m}\prod_{j=1}^{n} X_{i,j} \ge \prod_{i=1}^{m}\prod_{j=1}^{n} Y_{i,j}.$$

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2 Given positive integers n and k, $n > k^2 > 4$. In a $n \times n$ grid, a k-*group* is a set of k unit squares lying in different rows and different columns.
Determine the maximal possible N, such that one can choose N unit squares in the grid and color them, with the following condition holds: in any k-*group* from the colored N unit squares, there are two squares with the same color, and there are also two squares with different colors.

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3 Given positive integer n. Prove that for any integers $a_1, a_2, \cdots, a_n$, at least $\lceil \frac{n(n-6)}{19} \rceil$ numbers from the set $\{1, 2, \cdots, \frac{n(n-1)}{2}\}$ cannot be represented as $a_i - a_j (1 \le i, j \le n)$.

Test 1 Day 2

4 Let $f(x), g(x)$ be two polynomials with integer coefficients. It is known that for infinitely many prime p, there exist integer m_p such that

$$f(a) \equiv g(a + m_p) \pmod p$$

holds for all $a \in \mathbb{Z}$. Prove that there exists a rational number r such that

$$f(x) = g(x + r).$$

5 Given a triangle ABC, a circle Ω is tangent to AB, AC at B, C, respectively. Point D is the midpoint of AC, O is the circumcenter of triangle ABC. A circle Γ passing through A, C intersects the minor arc BC on Ω at P, and intersects AB at Q. It is known that the midpoint R of minor arc PQ satisfies that $CR \perp AB$. Ray PQ intersects line AC at L, M is the midpoint of AL, N is the midpoint of DR, and X is the projection of M onto ON. Prove that the circumcircle of triangle DNX passes through the center of Γ.

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6 Given positive integer n and r pairwise distinct primes $p_1, p_2, \cdots, p_r$. Initially, there are $(n+1)^r$ numbers written on the blackboard: $p_1^{i_1} p_2^{i_2} \cdots p_r^{i_r} (0 \le i_1, i_2, \cdots, i_r \le n)$.

Alice and Bob play a game by making a move by turns, with Alice going first. In Alice's round, she erases two numbers a, b (not necessarily different) and write $\gcd(a, b)$. In Bob's round, he erases two numbers a, b (not necessarily different) and write $\operatorname{lcm}(a, b)$. The game ends when only one number remains on the blackboard.

Determine the minimal possible M such that Alice could guarantee the remaining number no greater than M, regardless of Bob's move.

Test 2 Day 1

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1 A cyclic quadrilateral $ABCD$ has circumcircle Γ, and $AB + BC = AD + DC$. Let E be the midpoint of arc BCD, and $F (\neq C)$ be the antipode of A *wrt* Γ. Let I, J, K be the incenter of $\triangle ABC$, the A-excenter of $\triangle ABC$, the incenter of $\triangle BCD$, respectively.
Suppose that a point P satisfies $\triangle BIC \stackrel{+}{\sim} \triangle KPJ$. Prove that EK and PF intersect on Γ.

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2 Given positive integers n, k, $n \ge 2$. Find the minimum constant c satisfies the following assertion:
For any positive integer m and a kn-regular graph G with m vertices, one could color the vertices of G with n different colors, such that the number of monochrome edges is at most cm.

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3 Given positive integers a, b, c which are pairwise coprime. Let $f(n)$ denotes the number of the non-negative integer solution (x, y, z) to the equation

$$ax + by + cz = n.$$

Prove that there exists constants $\alpha, \beta, \gamma \in \mathbb{R}$ such that for any non-negative integer n,

$$|f(n) - \left(\alpha n^2 + \beta n + \gamma\right)| < \frac{1}{12}\left(a + b + c\right).$$

Test 2 Day 2

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4 Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that for all positive integers m, n with $m \ge n$,

$$f(m\varphi(n^3)) = f(m) \cdot \varphi(n^3).$$

Here $\varphi(n)$ denotes the number of positive integers coprime to n and not exceeding n.

5 Let n be a positive integer and $a_1, a_2, \dots a_{2n+1}$ be positive reals. For $k = 1, 2, \dots, 2n+1$, denote $b_k = \max_{0\le m\le n}\left(\frac{1}{2m+1}\sum_{i=k-m}^{k+m} a_i\right)$, where indices are taken modulo $2n+1$. Prove that the number of indices k satisfying $b_k \ge 1$ does not exceed $2\sum_{i=1}^{2n+1} a_i$.

6 Find the smallest positive real constant a, such that for any three points A, B, C on the unit circle, there exists an equilateral triangle PQR with side length a such that all of A, B, C lie on the interior or boundary of $\triangle PQR$.

Test 3 Day 1

1 Given positive integer $n \ge 5$ and a convex polygon P, namely $A_1A_2...A_n$. No diagonals of P are concurrent. Proof that it is possible to choose a point inside every quadrilateral $A_iA_jA_kA_l (1 \le i < j < k < l \le n)$ not on diagonals of P, such that the $\binom{n}{4}$ points chosen are distinct, and any segment connecting these points intersect with some diagonal of P.

2 Given distinct positive integer $a_1, a_2, \dots, a_{2020}$. For $n \ge 2021$, a_n is the smallest number different from $a_1, a_2, \dots, a_{n-1}$ which doesn't divide $a_{n-2020}...a_{n-2}a_{n-1}$. Proof that every number large enough appears in the sequence.

3 Determine the greatest real number C, such that for every positive integer $n \ge 2$, there exists $x_1, x_2, ..., x_n \in [-1, 1]$, so that

$$\prod_{1\le i<j\le n}(x_i - x_j) \ge C^{\frac{n(n-1)}{2}}$$

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– Day 2

4 Proof that

$$\sum_{m=1}^{n} 5^{\omega(m)} \le \sum_{k=1}^{n} \lfloor \frac{n}{k} \rfloor \tau(k)^2 \le \sum_{m=1}^{n} 5^{\Omega(m)}.$$

5 Determine all $f : R \to R$ such that

$$f(xf(y) + y^3) = yf(x) + f(y)^3$$

6 Proof that there exist constant λ, so that for any positive integer $m(\ge 2)$, and any lattice triangle T in the Cartesian coordinate plane, if T contains exactly one m-lattice point in its interior(not containing boundary), then T has area $\le \lambda m^3$.

PS. lattice triangles are triangles whose vertex are lattice points; m-lattice points are lattice points whose both coordinates are divisible by m.

Test 4 Day 1

1 Let $n(\geq 2)$ be a positive integer. Find the minimum m, so that there exists $x_{ij}(1 \leq i,j \leq n)$ satisfying:
(1)For every $1 \leq i,j \leq n, x_{ij} = max\{x_{i1}, x_{i2}, ..., x_{ij}\}$ or $x_{ij} = max\{x_{1j}, x_{2j}, ..., x_{ij}\}$.
(2)For every $1 \leq i \leq n$, there are at most m indices k with $x_{ik} = max\{x_{i1}, x_{i2}, ..., x_{ik}\}$.
(3)For every $1 \leq j \leq n$, there are at most m indices k with $x_{kj} = max\{x_{1j}, x_{2j}, ..., x_{kj}\}$.

2 Let triangle$ABC(AB < AC)$ with incenter I circumscribed in $\odot O$. Let M, N be midpoint of arc $\widehat{BAC}$ and $\widehat{BC}$, respectively. D lies on $\odot O$ so that $AD//BC$, and E is tangency point of A-excircle of $\triangle ABC$. Point F is in $\triangle ABC$ so that $FI//BC$ and $\angle BAF = \angle EAC$. Extend NF to meet $\odot O$ at G, and extend AG to meet line IF at L. Let line AF and DI meet at K. Proof that $ML \perp NK$.

3 Find all positive integer $n(\geq 2)$ and rational $\beta \in (0,1)$ satisfying the following:
There exist positive integers $a_1, a_2, ..., a_n$, such that for any set $I \subseteq \{1, 2, ..., n\}$ which contains at least two elements,

$$S(\sum_{i\in I} a_i) = \beta \sum_{i \in I} S(a_i).$$

where $S(n)$ denotes sum of digits of decimal representation of n.

– Day 2

4 Suppose $x_1, x_2, ..., x_{60} \in [-1, 1]$, find the maximum of

$$\sum_{i=1}^{60} x_i^2(x_{i+1} - x_{i-1}),$$

where $x_{i+60} = x_i$.

5 Find the smallest real α, such that for any convex polygon P with area 1, there exist a point M in the plane, such that the area of convex hull of $P \cup Q$ is at most α, where Q denotes the image of P under central symmetry with respect to M.

6 Let $n(\geq 2)$ be an integer. $2n^2$ contestants participate in a Chinese chess competition, where any two contestant play exactly once. There may be draws. It is known that
(1)If A wins B and B wins C, then A wins C.
(2)there are at most $\frac{n^3}{16}$ draws.
Proof that it is possible to choose n^2 contestants and label them $P_{ij}(1 \leq i,j \leq n)$, so that for any $i, j, i', j' \in \{1, 2, ..., n\}$, if $i < i'$, then P_{ij} wins $P_{i'j'}$.