Mathematical Olympiad 2021

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– Day 1

1 In convex quadrilateral $ABCD$, $\angle CAB = \angle BCD$. P lies on line BC such that $AP = PC$, Q lies on line AP such that AC and DQ are parallel, R is the point of intersection of lines AB and CD, and S is the point of intersection of lines AC and QR. Line AD meets the circumcircle of AQS again at T. Prove that AB and QT are parallel.

2 Let n be a positive integer. Show that there exists a one-to-one function $\sigma : \{1, 2, ..., n\} \to \{1, 2, ..., n\}$ such that

$$\sum_{k=1}^{n} \frac{k}{(k+\sigma(k))^2} < \frac{1}{2}.$$

3 Denote by $\mathbb{Q}^+$ the set of positive rational numbers. A function $f : \mathbb{Q}^+ \to \mathbb{Q}$ satisfies

$f(p) = 1$ for all primes p, and

$f(ab) = af(b) + bf(a)$ for all $a, b \in \mathbb{Q}^+$.

For which positive integers n does the equation $nf(c) = c$ have at least one solution c in $\mathbb{Q}^+$?

4 Determine the set of all polynomials $P(x)$ with real coefficients such that the set $\{P(n) | n \in \mathbb{Z}\}$ contains all integers, except possibly finitely many of them.

– Day 2

5 A positive integer is called *lucky* if it is divisible by 7, and the sum of its digits is also divisible by 7. Fix a positive integer n. Show that there exists some lucky integer l such that $|n - l| \leq 70$.

6 A certain country wishes to interconnect 2021 cities with flight routes, which are always two-way, in the following manner:

There is a way to travel between any two cities either via a direct flight or via a sequence of connecting flights.

For every pair (A, B) of cities that are connected by a direct flight, there is another city C such that (A, C) and (B, C) are connected by direct flights.

Show that at least 3030 flight routes are needed to satisfy the two requirements.

7 Let $a, b, c,$ and d be real numbers such that $a \geq b \geq c \geq d$ and

$$a + b + c + d = 13$$

$$a^2 + b^2 + c^2 + d^2 = 43.$$

Show that $ab \geq 3 + cd$.

8 In right triangle ABC, $\angle ACB = 90^\circ$ and $\tan A > \sqrt{2}$. M is the midpoint of AB, P is the foot of the altitude from C, and N is the midpoint of CP. Line AB meets the circumcircle of CNB again at Q. R lies on line BC such that QR and CP are parallel, S lies on ray CA past A such that $BR = RS$, and V lies on segment SP such that $AV = VP$. Line SP meets the circumcircle of CPB again at T. W lies on ray VA past A such that $2AW = ST$, and O is the circumcenter of SPM. Prove that lines OM and BW are perpendicular.