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1 $\log_8 2 = 0.2525$ in base 8 (to 4 places of decimals). Find $\log_8 4$ in base 8 (to 4 places of decimals).

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2 The Fibonacci sequence $f_1, f_2, f_3, \ldots$ is defined by $f_1 = f_2 = 1$, $f_{n+2} = f_{n+1} + f_n$. Find all n such that $f_n = n^2$.

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3 ABC is a triangle with $\angle A = 90^\circ$, $\angle B = 60^\circ$. The points A_1, B_1, C_1 on BC, CA, AB respectively are such that $A_1B_1C_1$ is equilateral and the perpendiculars (to BC at A_1, to CA at B_1 and to AB at C_1) meet at a point P inside the triangle. Find the ratios $PA_1 : PB_1 : PC_1$.

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4 p is a prime. Find all relatively prime positive integers m, n such that

$$\frac{m}{n} + \frac{1}{p^2} = \frac{m+p}{n+p}$$

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5 $f(x)$ is a polynomial of degree $2n$. Show that all polynomials $p(x)$, $q(x)$ of degree at most n such that $f(x)q(x) - p(x)$ has the form

$$\sum_{2n<k\leq 3n} (a^k + x^k)$$

have the same $p(x)/q(x)$.

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6 $f(x)$ is a real valued function defined for $x \geq 0$ such that $f(0) = 0$, $f(x+1) = f(x) + \sqrt{x}$ for all x, and

$$f(x) < \frac{1}{2} f\left(x - \frac{1}{2}\right) + \frac{1}{2} f\left(x + \frac{1}{2}\right) \quad \text{for all} \quad x \geq \frac{1}{2}$$

Show that $f\left(\frac{1}{2}\right)$ is uniquely determined.
