www.artofproblemsolving.com/community/c1974570
by parmenides51

1 Let $a_n = 2^{n-1}$ for $n > 0$. Let

$$b_n = \sum_{r+s \le n} a_r a_s$$

Find $b_n - b_{n-1}$, $b_n - 2b_{n-1}$ and b_n.

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2 Show that

$$1 - \frac{1}{k} \le n\left(\sqrt[n]{k} - 1\right) \le k - 1$$

for all positive integers n and positive reals k.

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3 Let $a_1 = 1$, $a_2 = 2^{a_1}$, $a_3 = 3^{a_2}$, $a_4 = 4^{a_3}$, $\dots$, $a_9 = 9^{a_8}$. Find the last two digits of a_9.

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4 Find all polynomials $p(x)$ such that $p(x^2) = p(x)^2$ for all x. Hence find all polynomials $q(x)$ such that

$$q\left(x^2 - 2x\right) = q\left(x - 2\right)^2$$

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5 Find the smallest positive real t such that

$$\begin{cases} x_1 + x_3 = 2tx_2 \\ x_2 + x_4 = 2tx_3 \\ x_3 + x_5 = 2tx_4 \end{cases}$$

has a solution x_1, x_2, x_3, x_4, x_5 in non-negative reals, not all zero.

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6 For which n can we find positive integers $a_1, a_2, \dots, a_n$ such that

$$a_1^2 + a_2^2 + \dots + a_n^2$$

is a square?
