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by parmenides51

1 Let a, b, c, d be real numbers such that $a > b > c > d \geq 0$ and $a + d = b + c$. Show that

$$x^a + x^d \geq x^b + x^c$$

for $x > 0$.

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2 Let s_m be the number $66 \cdots 6$ with m digits 6. Find

$$s_1 + s_2 + \cdots + s_n$$

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3 Two satellites are orbiting the earth in the equatorial plane at an altitude h above the surface. The distance between the satellites is always d, the diameter of the earth. For which h is there always a point on the equator at which the two satellites subtend an angle of $90°$?

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4 $b_0, b_1, b_2, \dots$ is a sequence of positive reals such that the sequence $b_0, cb_1, c^2b_2, c^3b_3, \dots$ is convex for all $c > 0$. (A sequence is convex if each term is at most the arithmetic mean of its two neighbors.) Show that $\ln b_0, \ln b_1, \ln b_2, \dots$ is convex.

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5 $k > 1$ is fixed. Show that for n sufficiently large for every partition of $\{1, 2, \dots, n\}$ into k disjoint subsets we can find $a \neq b$ such that a and b are in the same subset and $a + 1$ and $b + 1$ are in the same subset. What is the smallest n for which this is true?

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6 $p(x)$ is a polynomial of degree n with leading coefficient c, and $q(x)$ is a polynomial of degree m with leading coefficient c, such that

$$p(x)^2 = \left(x^2 - 1\right) q(x)^2 + 1$$

Show that $p'(x) = nq(x)$.
