www.artofproblemsolving.com/community/c1974575
by parmenides51

1 Solve the equations:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + \cdots + (n-1)x_{n-1} + nx_n = n \\ 2x_1 + 3x_2 + 4x_3 + \cdots + nx_{n-1} + x_n = n-1 \\ 3x_1 + 4x_2 + 5x_3 + \cdots + x_{n-1} + 2x_n = n-2 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ (n-1)x_1 + nx_2 + x_3 + \cdots + (n-3)x_{n-1} + (n-2)x_n = 2 \\ nx_1 + x_2 + 2x_3 + \cdots + (n-2)x_{n-1} + (n-1)x_n = 1 \end{cases}$$

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2 Find rational x in $(3,4)$ such that $\sqrt{x-3}$ and $\sqrt{x+1}$ are rational.

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3 Express

$$x^{13} + \frac{1}{x^{13}}$$

as a polynomial in $y = x + \frac{1}{x}$.

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4 $f(x)$ is continuous on the interval $[0,\pi]$ and satisfies

$$\int_0^\pi f(x)dx = 0, \qquad \int_0^\pi f(x)\cos x dx = 0$$

Show that $f(x)$ has at least two zeros in the interval $(0,\pi)$.

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5 Find the smallest positive integer a such that for some integers b, c the polynomial $ax^2 - bx + c$ has two distinct zeros in the interval $(0,1)$.

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6 Find the sharpest inequalities of the form $a \cdot AB < AG < b \cdot AB$ and $c \cdot AB < BG < d \cdot AB$ for all triangles ABC with centroid G such that $GA > GB > GC$.
