AMC 10 2021 Spring

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– A

– February 4th, 2021

1 What is the value of

$$(2^2-2)-(3^2-3)+(4^2-4)?$$

(A) 1 **(B)** 2 **(C)** 5 **(D)** 8 **(E)** 12

2 Portia's high school has 3 times as many students as Lara's high school. The two high schools have a total of 2600 students. How many students does Portia's high school have?

(A) 600 **(B)** 650 **(C)** 1950 **(D)** 2000 **(E)** 2050

3 The sum of two natural numbers is $17,402$. One of the two numbers is divisible by 10. If the units digit of that number is erased, the other number is obtained. What is the difference of these two numbers?

(A) $10,272$ **(B)** $11,700$ **(C)** $13,362$ **(D)** $14,238$ **(E)** $15,426$

4 A cart rolls down a hill, traveling 5 inches the first second and accelerating so that each successive 1-second time
interval, it travels 7 inches more than during the previous 1-second interval. The cart takes 30 seconds to reach the
bottom of the hill. How far, in inches, does it travel?

(A) 215 **(B)** 360 **(C)** 2992 **(D)** 3195 **(E)** 3242

5 The quiz scores of a class with $k>12$ students have a mean of 8. The mean of a collection of 12 of these quiz scores is 14. What is the mean of the remaining quiz scores in terms of k?

(A) $\frac{14-8}{k-12}$ **(B)** $\frac{8k-168}{k-12}$ **(C)** $\frac{14}{12}-\frac{k}{8}$ **(D)** $\frac{14(k-12)}{k^2}$ **(E)** $\frac{14(k-12)}{8k}$

6 Chantal and Jean start hiking from a trailhead toward a fire tower. Jean is wearing a heavy backpack and walks slower. Chantal starts walking at 4 miles per hour. Halfway to the tower, the trail becomes really steep, and Chantal slows down to 2 miles per hour. After reaching the tower, she immediately turns around and descends the steep part of the trail at 3 miles per hour.

She meets Jean at the halfway point. What was Jean's average speed, in miles per hour, until they meet?

(A) $\frac{12}{13}$ **(B)** 1 **(C)** $\frac{13}{12}$ **(D)** $\frac{24}{13}$ **(E)** 2

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7 Tom has a collection of 13 snakes, 4 of which are purple and 5 of which are happy. He observes that • all of his happy snakes can add • none of his purple snakes can subtract, and • all of his snakes that can't subtract also can't add

Which of these conclusions can be drawn about Tom's snakes?

(A) Purple snakes can add. **(B)** Purple snakes are happy. **(C)** Snakes that can add are purple. **(D)** Happy snakes are not purple. **(E)** Happy snakes can't subtract.

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8 When a student multiplied the number 66 by the repeating decimal,

$$1.\underline{ab}\underline{ab}\ldots = 1.\overline{ab},$$

where a and b are digits, he did not notice the notation and just multiplied 66 times $1.\underline{ab}$. Later he found that his answer is 0.5 less than the correct answer. What is the 2- digit integer $\underline{ab}$?

(A) 15 **(B)** 30 **(C)** 45 **(D)** 60 **(E)** 75

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9 What is the least possible value of $(xy-1)^2+(x+y)^2$ for real numbers x and y?

(A) 0 **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** 2

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10 Which of the following is equivalent to

$$(2+3)(2^2+3^2)(2^4+3^4)(2^8+3^8)(2^{16}+3^{16})(2^{32}+3^{32})(2^{64}+3^{64})?$$

(A) $3^{127}+2^{127}$ **(B)** $3^{127}+2^{127}+2\cdot3^{63}+3\cdot2^{63}$ **(C)** $3^{128}-2^{128}$ **(D)** $3^{128}+2^{128}$ **(E)** 5^{127}

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11 For which of the following integers b is the base-b number 2021_b-221_b not divisible by 3?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 7 **(E)** 8

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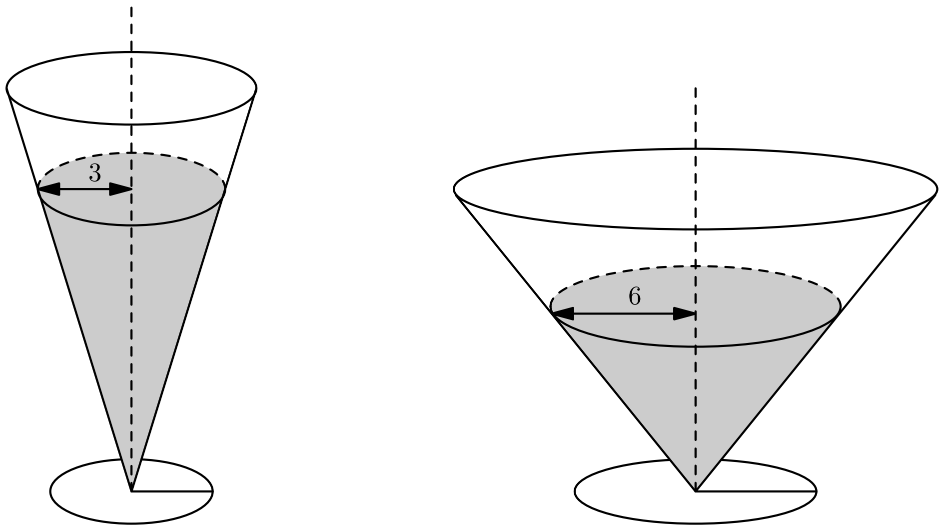
12 Two right circular cones with vertices facing down as shown in the figure below contain the same amount of liquid. The radii of the tops of the liquid surfaces are 3 cm and 6 cm. Into each cone is dropped a spherical marble of radius 1 cm, which sinks to the bottom and is completely submerged without spilling any liquid. What is the ratio of the rise of the liquid level in the narrow cone to the rise of the liquid level in the wide cone?

(A) $1:1$ **(B)** $47:43$ **(C)** $2:1$ **(D)** $40:13$ **(E)** $4:1$



13 What is the volume of tetrahedron $ABCD$ with edge lengths $AB = 2, AC = 3, AD = 4, BC = \sqrt{13}, BD = 2\sqrt{5}$, and $CD = 5$?

(A) 3 **(B)** $2\sqrt{3}$ **(C)** 4 **(D)** $3\sqrt{3}$ **(E)** 6

14 All the roots of polynomial $z^6 - 10z^5 + Az^4 + Bz^3 + Cz^2 + Dz + 16$ are positive integers. What is the value of B?

(A) -88 **(B)** -80 **(C)** -64 **(D)** -41 **(E)** -40

15 Values for $A, B, C,$ and D are to be selected from $\{1, 2, 3, 4, 5, 6\}$ without replacement (i.e. no two letters have the same value). How many ways are there to make such choices so that the two curves $y = Ax^2 + B$ and $y = Cx^2 + D$ intersect? (The order in which the curves are listed does not matter; for example, the choices $A = 3, B = 2, C = 4, D = 1$ is considered the same as the choices $A = 4, B = 1, C = 3, D = 2$.)

(A) 30 **(B)** 60 **(C)** 90 **(D)** 180 **(E)** 360

16 In the following list of numbers, the integer n appears n times in the list for $1 \leq n \leq 200$.

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots, 200, 200, \ldots, 200$$

What is the median of the numbers in this list?

(A) 100.5 **(B)** 134 **(C)** 142 **(D)** 150.5 **(E)** 167

17 Trapezoid $ABCD$ has $\overline{AB} \parallel \overline{CD}, BC = CD = 43$, and $\overline{AD} \perp \overline{BD}$. Let O be the intersection of the diagonals $\overline{AC}$ and $\overline{BD}$, and let P be the midpoint of $\overline{BD}$. GIven that $OP = 11$, the length

AD can be written in the form $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. What is $m+n$?

(A) 65 **(B)** 132 **(C)** 157 **(D)** 194 **(E)** 215

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18 Let f be a function defined on the set of positive rational numbers with the property that $f(a \cdot b) = f(a) + f(b)$ for all positive rational numbers a and b. Suppose that f also has the property that $f(p) = p$ for every prime number p. For which of the following numbers x is $f(x) < 0$?

(A) $\frac{17}{32}$ **(B)** $\frac{11}{16}$ **(C)** $\frac{7}{9}$ **(D)** $\frac{7}{6}$ **(E)** $\frac{25}{11}$

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19 The area of the region bounded by the graph of

$$x^2 + y^2 = 3|x-y| + 3|x+y|$$

is $m + n\pi$, where m and n are integers. What is $m+n$?

(A)18 **(B)**27 **(C)**36 **(D)**45 **(E)**54

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20 In how many ways can the sequence $1, 2, 3, 4, 5$ be arranged so that no three consecutive terms are increasing and no three consecutive terms are decreasing?

(A) 10 **(B)** 18 **(C)** 24 **(D)** 32 **(E)** 44

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21 Let $ABCDEF$ be an equiangular hexagon. The lines $AB, CD,$ and EF determine a triangle with area $192\sqrt{3}$, and the lines BC, DE, and FA determine a triangle with area $324\sqrt{3}$. The perimeter of hexagon $ABCDEF$ can be expressed as $m + n\sqrt{p}$, where $m, n,$ and p are positive integers and p is not divisible by the square of any prime. What is $m+n+p$?

(A) 47 **(B)** 52 **(C)** 55 **(D)** 58 **(E)** 63

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22 Hiram's algebra notes are 50 pages long and are printed on 25 sheets of paper; the first sheet contains pages 1 and 2, the second sheet contains pages 3 and 4, and so on. One day he leaves his notes on the table before leaving for lunch, and his roommate decides to borrow some pages from the middle of the notes. When Hiram comes back, he discovers that his roommate has taken a consecutive set of sheets from the notes and that the average (mean) of the page numbers on all remaining sheets is exactly 19. How many sheets were borrowed?

(A) 10 **(B)** 13 **(C)** 15 **(D)** 17 **(E)** 20

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23 Frieda the frog begins a sequence of hops on a 3×3 grid of squares, moving one square on each hop and choosing at random the direction of each hop up, down, left, or right. She does not hop diagonally. When the direction of a hop would take Frieda off the grid, she "wraps around" and jumps to the opposite edge. For example if Frieda begins in the center square and makes two hops "up", the first hop would place her in the top row middle square, and the second hop would cause Frieda to jump to the opposite edge, landing in the bottom row middle square. Suppose

Frieda starts from the center square, makes at most four hops at random, and stops hopping if she lands on a corner square. What is the probability that she reaches a corner square on one of the four hops?

(A) $\frac{9}{16}$ **(B)** $\frac{5}{8}$ **(C)** $\frac{3}{4}$ **(D)** $\frac{25}{32}$ **(E)** $\frac{13}{16}$

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24 The interior of a quadrilateral is bounded by the graphs of $(x+ay)^2 = 4a^2$ and $(ax-y)^2 = a^2$, where a is a positive real number. What is the area of this region in terms of a, valid for all $a > 0$?

(A) $\frac{8a^2}{(a+1)^2}$ **(B)** $\frac{4a}{a+1}$ **(C)** $\frac{8a}{a+1}$ **(D)** $\frac{8a^2}{a^2+1}$ **(E)** $\frac{8a}{a^2+1}$

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25 How many ways are there to place 3 indistinguishable red chips, 3 indistinguishable blue chips, and 3 indistinguishable green chips in the squares of a 3×3 grid so that no two chips of the same color are directly adjacent to each other, either vertically or horizontally?

(A) 12 **(B)** 18 **(C)** 24 **(D)** 30 **(E)** 36

– B

– February 10th, 2021

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1 How many integer values satisfy $|x| < 3\pi$?

(A) 9 **(B)** 10 **(C)** 18 **(D)** 19 **(E)** 20

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2 What is the value of $\sqrt{(3-2\sqrt{3})^2} + \sqrt{(3+2\sqrt{3})^2}$?

(A) 0 **(B)** $4\sqrt{3}-6$ **(C)** 6 **(D)** $4\sqrt{3}$ **(E)** $4\sqrt{3}+6$

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3 In an after-school program for juniors and seniors, there is a debate team with an equal number of students from each class on the team. among the 28 students in the program, 25

(A) 5 **(B)** 6 **(C)** 8 **(D)** 11 **(E)** $20.$

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4 At a math contest, 57 students are wearing blue shirts, and another 75 students are wearing yellow shirts. The 132 students are assigned into 66 pairs. In exactly 23 of these pairs, both students are wearing blue shirts. In how many pairs are both students wearing yellow shirts?

(A) 23 **(B)** 32 **(C)** 37 **(D)** 41 **(E)** 64

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5 The ages of Jonie's four cousins are distinct single-digit positive integers. Two of the cousins' ages multiplied together give 24, while the other two multiply to 30. What is the sum of the ages of Jonie's four cousins?

(A) 21 **(B)** 22 **(C)** 23 **(D)** 24 **(E)** 25

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6 Ms. Blackwell gives an exam to two classes. The mean of the scores of the students in the morning class is 84, and the afternoon class's mean score is 70. The ratio of the number of students in the morning class to the number of students in the afternoon class is $\frac{3}{4}$. What is the mean of the scores of all the students?

(A) 74 **(B)** 75 **(C)** 76 **(D)** 77 **(E)** 78

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7 In a plane, four circles with radii $1, 3, 5,$ and 7 are tangent to line l at the same point $A,$ but they may be on either side of l. Region S consists of all the points that lie inside exactly one of the four circles. What is the maximum possible area of region S?

(A) 24π **(B)** 32π **(C)** 64π **(D)** 65π **(E)** 84π

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8 Mr. Zhou places all the integers from 1 to 225 into a 15 by 15 grid. He places 1 in the middle square (eight row and eight column) and places the other numbers one by one clockwise, as shown in part in the diagram below. What is the sum of the greatest and the least number that appear in the second row from the top?

$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	21	22	23	24	25	$\cdots$
$\cdots$	20	7	8	9	10	$\cdots$
$\cdots$	19	6	1	2	11	$\cdots$
$\cdots$	18	5	4	3	12	$\cdots$
$\cdots$	17	16	15	14	13	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$

(A) 367 **(B)** 368 **(C)** 369 **(D)** 379 **(E)** 380

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9 The point $P(a,b)$ in the xy-plane is first rotated counterclockwise by 90° around the point $(1,5)$ and then reflected about the line $y=-x$. The image of P after these two transformations is at $(-6,3)$. What is $b-a$?

(A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

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10 An inverted cone with base radius 12 cm and height 18 cm is full of water. The water is poured into a tall cylinder whose horizontal base has a radius of 24 cm. What is the height in centimeters of the water in the cylinder?

(A) 1.5 **(B)** 3 **(C)** 4 **(D)** 4.5 **(E)** 6

11 Grandma has just finished baking a large rectangular pan of brownies. She is planning to make rectangular pieces of equal size and shape, with straight cuts parallel to the sides of the pan. Each cut must be made entirely across the pan. Grandma wants to make the same number of interior pieces as pieces along the perimeter of the pan. What is the greatest possible number of brownies she can produce?

(**A**) 24 (**B**) 30 (**C**) 48 (**D**) 60 (**E**) 64

12 Let $N = 34 \cdot 34 \cdot 63 \cdot 270$. What is the ratio of the sum of the odd divisors of N to the sum of the even divisors of N?

(A) $1:16$ **(B)** $1:15$ **(C)** $1:14$ **(D)** $1:8$ **(E)** $1:3$

13 Let n be a positive integer and d be a digit such that the value of the numeral $\underline{32d}$ in base n equals 263, and the value of the numeral 324 in base n equals the value of the numeral $\underline{11d1}$ in base six. What is $n+d$?

(A) 10 **(B)** 11 **(C)** 13 **(D)** 15 **(E)** 16

14 Three equally spaced parallel lines intersect a circle, creating three chords of lengths $38, 38,$ and 34. What is the distance between two adjacent parallel lines?

(A) $5\frac{1}{2}$ **(B)** 6 **(C)** $6\frac{1}{2}$ **(D)** 7 **(E)** $7\frac{1}{2}$

15 The real number x satisfies the equation $x+\frac{1}{x}=\sqrt{5}$. What is the value of $x^{11}-7x^7+x^3$?

(A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** 8

16 Call a positive integer an uphill integer if every digit is strictly greater than the previous digit. For example, $1357, 89,$ and 5 are all uphill integers, but $32, 1240,$ and 466 are not. How many uphill integers are divisible by 15?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

17 Ravon, Oscar, Aditi, Tyrone, and Kim play a card game. Each person is given 2 cards out of a set of 10 cards numbered $1, 2, 3, \dots, 10$. The score of a player is the sum of the numbers of their cards. The scores of the players are as follows: Ravon–11, Oscar–4, Aditi–7, Tyrone–16, Kim–17. Which of the following statements is true?

(A) Ravon was given card 3.

(B) Aditi was given card 3.

(C) Ravon was given card 4.

(D) Aditi was given card 4.

(E) Tyrone was given card 7.

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18 A fair 6-sided die is repeatedly rolled until an odd number appears. What is the probability that every even number appears at least once before the first occurrence of an odd number?

(A) $\frac{1}{120}$ **(B)** $\frac{1}{32}$ **(C)** $\frac{1}{20}$ **(D)** $\frac{3}{20}$ **(E)** $\frac{1}{6}$

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19 Suppose that S is a finite set of positive integers. If the greatest integer in S is removed from S, then the average value (arithmetic mean) of the integers remaining is 32. If the least integer is S is *also* removed, then the average value of the integers remaining is 35. If the greatest integer is then returned to the set, the average value of the integers rises to 40. The greatest integer in the original set S is 72 greater than the least integer in S. What is the average value of all the integers in the set S?

(A) 36.2 **(B)** 36.4 **(C)** 36.6 **(D)** 36.8 **(E)** 37

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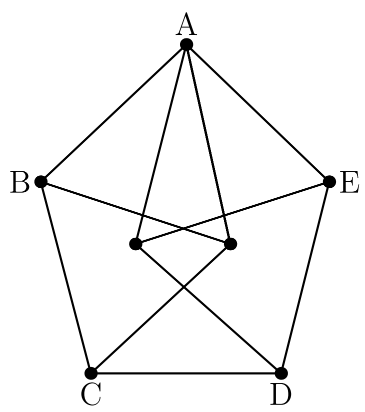
20 The figure below is constructed from 11 line segments, each of which has length 2. The area of pentagon $ABCDE$ can be written as $\sqrt{m} + \sqrt{n}$, where m and n are positive integers. What is $m + n$?



(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

Proposed by **djmathman**

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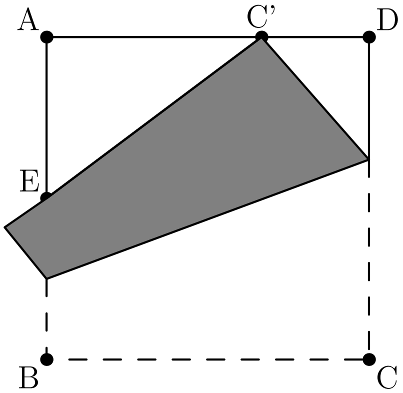
21 A square piece of paper has side length 1 and vertices $A, B, C,$ and D in that order. As shown in the figure, the paper is folded so that vertex C meets edge $\overline{AD}$ at point C', and edge $\overline{BC}$ intersects edge $\overline{AB}$ at point E. Suppose that $C'D = \frac{1}{3}$. What is the perimeter of $\triangle AEC'$?



(A) 2 **(B)** $1+\frac{2}{3}\sqrt{3}$ **(C)** $\frac{13}{6}$ **(D)** $1+\frac{3}{4}\sqrt{3}$ **(E)** $\frac{7}{3}$

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22 Ang, Ben, and Jasmin each have 5 blocks, colored red, blue, yellow, white, and green; and there are 5 empty boxes. Each of the people randomly and independently of the other two people places one of their blocks into each box. The probability that at least one box receives 3 blocks all of the same color is $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m+n$?

(A) 47 **(B)** 94 **(C)** 227 **(D)** 471 **(E)** 542

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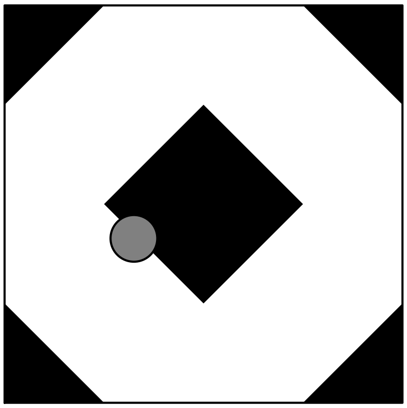
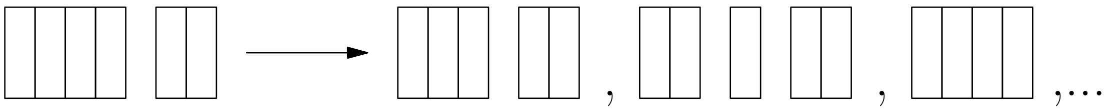
23 A square with side length 8 is colored white except for 4 black isosceles right triangular regions with legs of length 2 in each corner of the square and a black diamond with side length $2\sqrt{2}$ in the center of the square, as shown in the diagram. A circular coin with diameter 1 is dropped onto the square and lands in a random location where the coin is completely contained within the square, The probability that the coin will cover part of the black region of the square can be written as $\frac{1}{196}(a+b\sqrt{2}+\pi)$, where a and b are positive integers. What is $a+b$?



(A) 64 **(B)** 66 **(C)** 68 **(D)** 70 **(E)** 72

24 Arjun and Beth play a game in which they take turns removing one brick or two adjacent bricks from one "wall" among a set of several walls of bricks, with gaps possibly creating new walls. The walls are one brick tall. For example, a set of walls of sizes 4 and 2 can be changed into any of the following by one move: $(3,2),(2,1,2),(4),(4,1),(2,2),$ or $(1,1,2)$.



Arjun plays first, and the player who removes the last brick wins. For which starting configuration is there a strategy that guarantees a win for Beth?

(A) $(6,1,1)$ **(B)** $(6,2,1)$ **(C)** $(6,2,2)$ **(D)** $(6,3,1)$ **(E)** $(6,3,2)$

25 Let S be the set of lattice points in the coordinate plane, both of whose coordinates are integers between 1 and $30,$ inclusive. Exactly 300 points in S lie on or below a line with equation $y=mx.$ The possible values of m lie in an interval of length $\frac{a}{b},$ where a and b are relatively prime positive integers. What is $a+b?$

(A) 31 **(B)** 47 **(C)** 62 **(D)** 72 **(E)** 85

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