www.artofproblemsolving.com/community/c1975449
by parmenides51

1 Show that the polynomial $x^6 - x^5 + x^4 - x^3 + x^2 - x + \frac{3}{4}$ has no real zeroes.

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2 The diagonals AC and BD of a quadrilateral $ABCD$ intersect at O. If S_1 and S_2 are the areas of triangles AOB and COD and S that of $ABCD$, show that $\sqrt{S_1} + \sqrt{S_2} \le \sqrt{S}$. Prove that equality holds if and only if AB and CD are parallel.

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3 Let $N \ge 3$ be a positive integer. For every pair (a, b) of integers with $1 \le a < b \le N$ consider the quotient $q = b/a$. Show that the pairs with $q < 2$ are equally numbered as those with $q > 2$.

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4 Prove that $x = y = z = 1$ is the only positive solution of the system

$$\begin{cases} x + y^2 + z^3 = 3 \\ y + z^2 + x^3 = 3 \\ z + x^2 + y^3 = 3 \end{cases}$$

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5 In the arrangement of pn real numbers below, the difference between the greatest and smallest numbers in each row is at most d, $d > 0$.

$$\begin{matrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ a_{n1} & a_{n2} & ... & a_{nn} \end{matrix}$$

Prove that, when the numbers in each column are rearranged in decreasing order, the difference between the greatest and smallest numbers in each row will still be at most d.

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6 The interval $[0, 1]$ is covered by a finite number of intervals. Show that one can choose a number of these intervals which are pairwise disjoint and have the total length at least $1/2$.

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