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by parmenides51

1 Let $a > b > c$ be sides of a triangle and h_a, h_b, h_c be the corresponding altitudes.
Prove that $a + h_a > b + h_b > c + h_c$.

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2 Six ducklings swim on the surface of a pond, which is in the shape of a circle with radius 5 m. Show that at every moment, two of the ducklings swim on the distance of at most 5 m from each other.

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3 Show that if $x_1 + x_2 + x_3 = 0$ for real numbers x_1, x_2, x_3, then $x_1x_2 + x_2x_3 + x_3x_1 \leq 0$.

Find all $n \geq 4$ for which $x_1 + x_2 + ... + x_n = 0$ implies $x_1x_2 + x_2x_3 + ... + x_{n-1}x_n + x_nx_1 \leq 0$.

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4 A polynomial $P(x)$ of degree 3 has three distinct real roots.
Find the number of real roots of the equation $P'(x)^2 - 2P(x)P''(x) = 0$.

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5 Show that there exists a constant $a > 1$ such that, for any positive integers m and n, $\frac{m}{n} < \sqrt{7}$ implies that

$$7 - \frac{m^2}{n^2} \geq \frac{a}{n^2}.$$

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6 The sequence (a_n) is defined by $a_1 = 1$ and $a_{n+1} = \sqrt{a_n^2 + \frac{1}{a_n}}$ for $n \geq 1$.
Prove that there exists a such that $\frac{1}{2} \leq \frac{a_n}{n^a} \leq 2$ for $n \geq 1$.
