OMMC Year 1 2020-2021 problems

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by squareman, Catsaway, billert

– **Round 1** February 2021

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1 Find the remainder when
$$20^{20}+21^{21}-21^{20}-20^{21}$$
is divided by 100.

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2 There are a family of 5 siblings. They have a pile of at least 2 candies and are trying to split them up
amongst themselves. If the 2 oldest siblings share the candy equally, they will have 1 piece of candy left over.
If the 3 oldest siblings share the candy equally, they will also have 1 piece of candy left over. If all 5 siblings
share the candy equally, they will also have 1 piece left over. What is the minimum amount of candy required
for this to be true?

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3 Define $f(x)$ as $\frac{x^2-x-2}{x^2+x-6}$. $f(f(f(f(1))))$ can be expressed as $\frac{p}{q}$ for relatively prime positive integers p,q. Find $10p+q$.

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4 Robert tiles a 420×420 square grid completely with 1×2 blocks, then notices that the two diagonals of the grid pass through a total of n blocks. Find the sum of all possible values of n.

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5 Two points A, B are randomly chosen on a circle with radius 100. For a positive integer x, denote $P(x)$ as the probability that the length of AB is less than x. Find the minimum possible integer value of x such that P$(x) > \frac{2}{3}$.

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6 Jason and Jared take turns placing blocks within a game board with dimensions 3×300, with Jason going first, such that no two blocks can overlap. The player who cannot place a block within the boundaries loses. Jason can only place 2×100 blocks, and Jared can only place $2 \times n$ blocks where n is some positive integer greater than 3. Find the smallest integer value of n that still guarantees Jason a win (given both players are playing optimally).

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7 Derek fills a square 10 by 10 grid with 50 1s and 50 2s. He takes the product of the numbers in each of the 10 rows. He takes the product of the numbers in each of the 10 columns. He then sums these 20 products up to get an integer N. Find the minimum possible value of N.

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8 The function $g\left(x\right)$ is defined as $\sqrt{\dfrac{x}{2}}$ for all positive x.

$$g\left(g\left(g\left(g\left(g\left(\frac{1}{2}\right)+1\right)+1\right)+1\right)+1\right)$$

can be expressed as $\cos(b)$ using degrees, where $0^\circ < b < 90^\circ$ and $b = p/q$ for some relatively prime positive integers p, q. Find $p+q$.

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9 The difference between the maximum and minimum values of

$$2\cos 2x + 7\sin x$$

over the real numbers equals $\frac{p}{q}$ for relatively prime positive integers p, q. Find $p+q$.

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10 How many ways are there to arrange the numbers 1 through 8 into a 2 by 4 grid such that the sum of the numbers in each of the two rows are all multiples of 6, and the sum of the numbers in each of the four columns are all multiples of 3?

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11 In equilateral triangle XYZ with side length 10, define points A, B on XY, points C, D on YZ, and points E, F on ZX such that $ABDE$ and $ACEF$ are rectangles, $XA < XB$, $YC < YD$, and $ZE < ZF$. The area of hexagon $ABCDEF$ can be written as $\sqrt{x}$ for some positive integer x. Find x.

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12 Let $P(x) = x^3 + 8x^2 - x + 3$ and let the roots of P be a, b, and c. The roots of a monic polynomial $Q(x)$ are $ab - c^2, ac - b^2, bc - a^2$. Find $Q(-1)$.

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13 Find the number of nonnegative integers $n < 29$ such that there exists positive integers x, y where

$$x^2 + 5xy - y^2$$

has remainder n when divided by 29.

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14 There exist positive integers N, M such that N's remainders modulo the four integers $6, 36, 216$, and M form an increasing nonzero geometric sequence in that order. Find the smallest possible value of M.

– **Round 2** March 2021

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15 A point X exactly $\sqrt{2} - \frac{\sqrt{6}}{3}$ away from the origin is chosen randomly. A point Y less than 4 away from the origin is chosen randomly. The probability that a point Z less than 2 away from the origin exists such that $\triangle XYZ$ is an equilateral triangle can be expressed as $\frac{a\pi + b}{c\pi}$ for some positive integers a, b, c with a and c relatively prime. Find $a+b+c$.

1 There are 20 people in a particular social network. Each person follows exactly 2 others in this network, and also has 2 people following them as well. What is the maximum possible number of people that can be placed into a subset of the network such that no one in this subset follows someone else in the subset?

2 Sequences a_n and b_n are defined for all positive integers n such that $a_1 = 5, b_1 = 7,$

$$a_{n+1} = \frac{\sqrt{(a_n + b_n - 1)^2 + (a_n - b_n + 1)^2}}{2},$$

and

$$b_{n+1} = \frac{\sqrt{(a_n + b_n + 1)^2 + (a_n - b_n - 1)^2}}{2}.$$

How many integers n from 1 to 1000 satisfy the property that a_n, b_n form the legs of a right triangle with a hypotenuse that has integer length?

3 The intersection of two squares with perimeter 8 is a rectangle with diagonal length 1. Given that the distance between the centers of the two squares is 2, the perimeter of the rectangle can be expressed as P. Find $10P$.

4 The sum

$$\frac{1^2 - 2}{1!} + \frac{2^2 - 2}{2!} + \frac{3^2 - 2}{3!} + \cdots + \frac{2021^2 - 2}{2021!}$$

can be expressed as a rational number N. Find the last 3 digits of $2021! \cdot N$.

5 Let N be an 3 digit integer in base 10 such that the sum of its digits in base 4 is half the sum of its digits in base 8. In base 10, find the largest possible value of N.

6 In square $ABCD$ with $AB = 10$, point P, Q are chosen on side CD and AD respectively such that $BQ \perp AP$, and R lies on CD such that $RQ \parallel PA$. BC and AP intersect at X, and XQ intersects the circumcircle of PQD at Y. Given that $\angle PYR = 105^\circ$, AQ can be expressed in simplest radical form as $b\sqrt{c} - a$ where a, b, c are positive integers. Find $a + b + c$.

7 An infinitely large grid is filled such that each grid square contains exactly one of the digits $\{1, 2, 3, 4\}$, each digit appears at least once, and the digit in each grid square equals the digit located 5 squares above it as well as the digit located 5 squares to the right. A group of 4 horizontally adjacent digits or 4 vertically adjacent digits is chosen randomly, and depending on its orientation is read left to right or top to bottom to form an 4-digit integer. The expected value of this integer is also a 4-digit integer N. Given this, find the last three digits of the sum of all possible values of N.

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8 Let triangle MAD be inscribed in circle O with diameter 85 such that $MA = 68$ and $DA = 40$. The altitudes from M, D to sides AD and MA, respectively, intersect the tangent to circle O at A at X and Y respectively. $XA \times YA$ can be expressed as $\frac{a}{b}$, where a and b are relatively prime positive integers. Find $a + b$.

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9 The infinite sequence of integers $a_1, a_2, \cdots$ is defined recursively as follows: $a_1 = 3$, $a_2 = 7$, and a_n equals the alternating sum

$$a_1 - 2a_2 + 3a_3 - 4a_4 + \cdots (-1)^n \cdot (n-1)a_{n-1}$$

for all $n > 2$. Let a_x be the smallest positive multiple of 1090 appearing in this sequence. Find the remainder of a_x when divided by 113.

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10 An *indivisible tiling* is a tiling of an $m \times n$ rectangular grid using only rectangles with a width and/or length of 1, such that nowhere in the tiling is a smaller complete tiling of a rectangle with more than 1 tile. Find the smallest integer a such that an indivisible tiling of an $a \times a$ square may contain exactly 2021 1×1 tiles.

– **Round 3** February 2021

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1 A man rows at a speed of 2 mph in still water. He set out on a trip towards a spot 2 miles downstream. He rowed with the current until he was halfway there, then turned back and rowed against the current for 15 minutes. Then, he turned around again and rowed with the current until he reached his destination. The entire trip took him 70 minutes. The speed of the current can be represented as $\frac{p}{q}$ mph where p, q are relatively prime positive integers. Find $10p + q$.

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2 The function $f(x)$ is defined on the reals such that

$$f\left(\frac{1-4x}{4-x}\right) = 4 - xf(x)$$

for all $x \neq 4$. There exists two distinct real numbers $a, b \neq 4$ such that $f(a) = f(b) = \frac{5}{2}$. $a + b$ can be represented as $\frac{p}{q}$ where p, q are relatively prime positive integers. Find $10p + q$.

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3 Two real numbers x, y are chosen randomly and independently on the interval $(1, r)$ where r is some real number between 1024 and 2048. Let P be the probability that $\lfloor \log_2 x \rfloor > \lfloor \log_2 y \rfloor$. The value of P is maximized when $r = \frac{p}{q}$ where p, q are relatively prime positive integers. Find $p + q$.

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4 In 3-dimensional space, two spheres centered at points O_1 and O_2 with radii 13 and 20 respectively intersect in a circle. Points A, B, C lie on that circle, and lines O_1A and O_1B intersect sphere O_2 at points D and E respectively. Given that $O_1O_2 = AC = BC = 21$, DE can be expressed as $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers. Find $a + b + c$.

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5 How many nonempty subsets of $1, 2, \dots, 15$ are there such that the sum of the squares of each subset is a multiple of 5?

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6 Find the minimum possible value of

$$\left(\sqrt{x^2+4}+\sqrt{x^2+7\sqrt{3}x+49}\right)^2$$

over all real numbers.

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7 Find the number of ordered triples of integers (a, b, c) such that

$$a^2+b^2+c^2-ab-bc-ca-1 \le 4042b-2021a-2021c-2021^2$$

and $|a|, |b|, |c| \le 2021$.

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8 Triangle ABC has circumcircle ω. The angle bisectors of $\angle A$ and $\angle B$ intersect ω at points D and E respectively. DE intersects BC and AC at X and Y respectively. Given $DX = 7$, $XY = 8$ and $YE = 9$, the area of $\triangle ABC$ can be written as $\frac{a\sqrt{b}}{c}$ where a, b, c are positive integers, $\gcd(a, c) = 1$, and b is square free. Find $a + b + c$.

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9 There is a 4×4 array of integers A, all initially equal to 0. An operation may be performed on the array for any row or column such that every number in that row or column has 1 added to it, and then is replaced with its remainder modulo 3. Given a random 4×4 array of integers between 0 and 2 not identical to A, the probability that it can be reached through a series of operations on A is $\frac{p}{q}$, where p, q are relatively prime positive integers. Find p.

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10 Positive integers a, b, c exist such that $a + b + c + 1$, $a^2 + b^2 + c^2 + 1$, $a^3 + b^3 + c^3 + 1$, and $a^4 + b^4 + c^4 + 7459$ are all multiples of p for some prime p. Find the sum of all possible values of p less than 1000.
