Canadian Mathematical Olympiad Qualification Repechage 2021

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by Audiophile

1 Determine all real polynomials p such that $p(x+p(x)) = x^2 p(x)$ for all x.

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2 Determine all integer solutions to the system of equations:

$$xy + yz + zx = -4$$
$$x^2 + y^2 + z^2 = 24$$
$$x^3 + y^3 + z^3 + 3xyz = 16$$

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3 $ABCDE$ is a regular pentagon. Two circles C_1 and C_2 are drawn through B with centers A and C respectively. Let the other intersection of C_1 and C_2 be P. The circle with center P which passes through E and D intersects C_2 at X and AE at Y. Prove that $AX = AY$.

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4 Let O be the centre of the circumcircle of triangle ABC and let I be the centre of the incircle of triangle ABC. A line passing through the point I is perpendicular to the line IO and passes through the incircle at points P and Q. Prove that the diameter of the circumcircle is equal to the perimeter of triangle OPQ.

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5 Alphonse and Beryl are playing a game. The game starts with two rectangles with integer side lengths. The players alternate turns, with Alphonse going first. On their turn, a player chooses one rectangle, and makes a cut parallel to a side, cutting the rectangle into two pieces, each of which has integer side lengths. The player then discards one of the three rectangles (either the one they did not cut, or one of the two pieces they cut) leaving two rectangles for the other player. A player loses if they cannot cut a rectangle.

Determine who wins each of the following games:

(a) The starting rectangles are 1×2020 and 2×4040.
(b) The starting rectangles are 100×100 and 100×500.

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6 Show that $(w, x, y, z) = (0, 0, 0, 0)$ is the only integer solution to the equation

$$w^2 + 11x^2 - 8y^2 - 12yz - 10z^2 = 0$$

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7 If A, B and C are real angles such that

$$\cos(B-C) + \cos(C-A) + \cos(A-B) = -3/2,$$

find

$$\cos(A) + \cos(B) + \cos(C)$$

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8 King Radford of Peiza is hosting a banquet in his palace. The King has an enormous circular table with 2021 chairs around it. At The King's birthday celebration, he is sitting in his throne (one of the 2021 chairs) and the other 2020 chairs are filled with guests, with the shortest guest sitting to the King's left and the remaining guests seated in increasing order of height from there around the table. The King announces that everybody else must get up from their chairs, run around the table, and sit back down in some chair. After doing this, The King notices that the person seated to his left is different from the person who was previously seated to his left. Each other person at the table also notices that the person sitting to their left is different.

Find a closed form expression for the number of ways the people could be sitting around the table at the end. You may use the notation $D_n,$ the number of derangements of a set of size n, as part of your expression.
