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– Day 1

1 Define the sequence (a_n) as $a_1 = 1$, $a_{2n} = a_n$ and $a_{2n+1} = a_n + 1$ for all $n \geq 1$.

a) Find all positive integers n such that $a_{kn} = a_n$ for all integers $1 \leq k \leq n$.
b) Prove that there exist infinitely many positive integers m such that $a_{km} \geq a_m$ for all positive integers k.

2 In a board of 2021×2021 grids, we pick k unit squares such that every picked square shares vertice(s) with at most 1 other picked square. Determine the maximum of k.

3 Let ABC be a triangle and N be a point that differs from A, B, C. Let A_b be the reflection of A through NB, and B_a be the reflection of B through NA. Similarly, we define B_c, C_b, A_c, C_a. Let m_a be the line through N and perpendicular to B_cC_b. Define similarly m_b, m_c.

a) Assume that N is the orthocenter of $\triangle ABC$, show that the respective reflection of m_a, m_b, m_c through the bisector of angles $\angle BNC, \angle CNA, \angle ANB$ are the same line.
b) Assume that N is the nine-point center of $\triangle ABC$, show that the respective reflection of m_a, m_b, m_c through BC, CA, AB concur.

– Day 2

4 Let a, b, c are non-negative numbers such that

$$2(a^2 + b^2 + c^2) + 3(ab + bc + ca) = 5(a + b + c)$$

then prove that $4(a^2 + b^2 + c^2) + 2(ab + bc + ca) + 7abc \leq 25$

5 Given a fixed circle (O) and two fixed points B, C on that circle, let A be a moving point on (O) such that $\triangle ABC$ is acute and scalene. Let I be the midpoint of BC and let AD, BE, CF be the three heights of $\triangle ABC$. In two rays $\overrightarrow{FA}, \overrightarrow{EA}$, we pick respectively M, N such that $FM = CE, EN = BF$. Let L be the intersection of MN and EF, and let $G \neq L$ be the second intersection of (LEN) and (LFM).

a) Show that the circle (MNG) always goes through a fixed point.

b) Let AD intersects (O) at $K \neq A$. In the tangent line through D of (DKI), we pick P, Q such that $GP \parallel AB, GQ \parallel AC$. Let T be the center of (GPQ). Show that GT always goes through a fixed point.

6 Let $n \geq 3$ be a positive integers and p be a prime number such that $p > 6^{n-1} - 2^n + 1$. Let S be the set of n positive integers with different residues modulo p. Show that there exists a positive integer c such that there are exactly two ordered triples $(x, y, z) \in S^3$ with distinct elements, such that $x - y + z - c$ is divisible by p.
