Ukraine National Mathematical Olympiad 2021

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by parmenides51

– grade X

– day 1

1 Alexey and Bogdan play a game with two piles of stones. In the beginning, one of the piles contains 2021 stones, and the second is empty. In one move, each of the guys has to pick up an even number of stones (more than zero) from an arbitrary pile, then transfer half of the stones taken to another pile, and the other half - to remove from the game. Loses the one who cannot make a move. Who will win this game if both strive to win, and Bogdan begins?

(Oleksii Masalitin)

2 Denote by $P^{(n)}$ the set of all polynomials of degree n the coefficients of which is a permutation of the set of numbers $\{2^0, 2^1, ..., 2^n\}$. Find all pairs of natural numbers (k, d) for which there exists a n such that for any polynomial $p \in P^{(n)}$, number $P(k)$ is divisible by the number d.

(Oleksii Masalitin)

3 For arbitrary positive reals $a \geq b \geq c$ prove the inequality:

$$\frac{a^2+b^2}{a+b}+\frac{a^2+c^2}{a+c}+\frac{c^2+b^2}{c+b} \geq (a+b+c)+\frac{(a-c)^2}{a+b+c}$$

(Anton Trygub)

4 Let O, I, H be the circumcenter, the incenter, and the orthocenter of $\triangle ABC$. The lines AI and AH intersect the circumcircle of $\triangle ABC$ for the second time at D and E, respectively. Prove that if $OI \parallel BC$, then the circumcenter of $\triangle OIH$ lies on DE.

(Fedir Yudin)

– day 2

5 Find all sets of $n \geq 2$ consecutive integers $\{a+1, a+2, ..., a+n\}$ where $a \in Z$, in which one of the numbers is equal to the sum of all the others.

(Bogdan Rublev)

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6 Circles w_1 and w_2 intersect at points P and Q and touch a circle w with center at point O internally at points A and B, respectively. It is known that the points A, B and Q lie on one line. Prove that the point O lies on the external bisector $\angle APB$.

(Nazar Serdyuk)

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7 The sequence $a_1, a_2, ..., a_{2n}$ of integers is such that each number occurs in no more than n times. Prove that there are two strictly increasing sequences of indices $b_1, b_2, ..., b_n$ and $c_1, c_2, ..., c_n$ are such that every positive integer from the set $\{1, 2, ..., 2n\}$ occurs exactly in one of these two sequences, and for each $1 \leq i \leq n$ is true the condition $a_{b_i} \neq a_{c_i}$
.
(Anton Trygub)

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8 Given a natural number n. Prove that you can choose $\phi(n) + 1$ (not necessarily different) divisors n with the sum n.

Here $\phi(n)$ denotes the number of natural numbers less than n that are coprime with n.

(Fedir Yudin)

– grade XI

– day 1

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1 It is known that for some integers $a_{2021}, a_{2020}, ..., a_1, a_0$ the expression

$$a_{2021}n^{2021} + a_{2020}n^{2020} + ... + a_1 n + a_0$$

is divisible by 2021 for any arbitrary integer n. Is it required that each of the numbers $a_{2021}, a_{2020}, ..., a_1, a_0$ also divisible by 2021?

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2 Find all natural numbers $n \geq 3$ for which in an arbitrary n-gon one can choose 3 vertices dividing its boundary into three parts, the lengths of which can be the lengths of the sides of some triangle.

(Fedir Yudin)

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3 same as X p4

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4 Find all the following functions $f : R \to R$, which for arbitrary valid x, y holds equality:

$$f(xf(x + y)) + f((x + y)f(y)) = (x + y)^2$$

(Vadym Koval)

– day 2

5 Are there natural numbers (m, n, k) that satisfy the equation $m^m + n^n = k^k$?

6 The altitudes AA_1, BB_1 and CC_1 were drawn in the triangle ABC. Point K is a projection of point B on A_1C_1. Prove that the symmmedian $\triangle ABC$ from the vertex B divides the segment B_1K in half.

(Anton Trygub)

7 same as XI p8

8 There are 101 not necessarily different weights, each of which weighs an integer number of grams from 1 g to 2020 g. It is known that at any division of these weights into two heaps, the total weight of at least one of the piles is no more than 2020. What is the largest number of grams can weigh all 101 weights?

(Bogdan Rublev)

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